Journal of Functional Analysis 270 (2016) 1001–1030

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Gabor frames for quasicrystals, K-theory, and twisted gap labeling

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ARTICLE INFO

Article history:
Received 26 November 2014
Accepted 18 November 2015
Available online 29 November 2015
Communicated by D. Voiculescu

Keywords:
Quasicrystals
Gabor analysis
Gap labeling

ABSTRACT

We study the connection between Gabor frames for quasicrystals, the topology of the hull of a quasicrystal Λ, and the K-theory of an associated twisted groupoid algebra. In particular, we construct a finitely generated projective module over this algebra, and any multiwindow Gabor frame for Λ can be used to construct an idempotent representing this module in K-theory. For lattice subsets in dimension two, this allows us to prove a twisted version of Bellissard's gap labeling theorem.

1. Introduction

The first examples of mathematical quasicrystals were studied by Meyer in [21]. Meyer thought of quasicrystals as generalizations of lattices which retained enough lattice-like structure to be useful for studying sampling problems in harmonic analysis. In another direction, the mathematical theory of quasicrystals began developing rapidly after real, physical quasicrystals were discovered by Shechtman et al. [27]. This led to the study of the topological dynamics of the hull Ω_Λ of a quasicrystal Λ, which are directly related to a variety of questions and constructions in symbolic dynamics (see [1] and [25] for an introduction). Bellissard's gap labeling conjecture provides a clear connection between

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http://dx.doi.org/10.1016/j.jfa.2015.11.011

the mathematics and physics [5]. While Bellissard's work demonstrates the value of topology and dynamics in studying the physics of quasicrystals, little has been done to integrate Meyer's original vision into this picture. The goal of the present paper is to show one avenue by which these strands of research can be connected. Namely, we will show how Gabor frames for a quasicrystal can be made compatible with its topological dynamics, and we use this connection to prove a twisted version of Bellissard's gap labeling conjecture for two-dimensional quasicrystals.

To elaborate, we will begin by describing Bellissard's gap labeling conjecture in detail. Given a quasicrystal Λ, we can imagine a material with an electron at each point in Λ. In order to analyze electron interactions in Λ, one studies a Schrödinger operator of the form

$$H_\Lambda = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{A}\right)^2 + \sum_{y\in\Lambda} v(\cdot - y)$$

acting on $L^2(\mathbb{R}^d)$, where v is a suitable potential ([3], Section 2.7). The vector potential $\vec{A}$ models the effect of a constant, uniform magnetic field. With appropriate boundary conditions, it is possible to restrict H_Λ to an operator $H_{\Lambda,R}$ on $L^2(C_R(0))$ where $C_R(0)$ is the closed cube of side length R. Then we can define the integrated density of states (IDOS)

$$\mathcal{N}(E) = \lim_{R\to\infty} \frac{1}{|C_R(0)|} |\{E' \in \mathrm{Sp}(H_{\Lambda,R}) \mid E' \le E\}|$$

which is used to express thermodynamical properties such as the heat capacity.

The IDOS can also be expressed using the language of operator algebras. There are natural C^* and Von Neumann (VN) algebras related to H_Λ which are generated by the resolvent of H_Λ. Essentially, these operator algebras are the same as the twisted groupoid algebras $\mathcal{A}_\theta = C^*(R_\Lambda, \theta)$ described in Section 2.2, where the cocycle θ is determined by the magnetic field and is the restriction of a cocycle on $\mathbb{R}^{2d}$ (see [3] and [5] for details). The algebra $\mathcal{A}_\theta$ is simple and has a unique normalized trace Tr. The associated VN algebra will be denoted by $\mathcal{A}_\theta''$. The spectral projection of H_Λ onto $(-\infty, E]$ is denoted by $\chi_E(H_\Lambda)$, and lies in $\mathcal{A}_\theta''$. This allows us to describe the IDOS using the trace on $\mathcal{A}_\theta''$ as

$$\mathcal{N}(E) = Tr(\chi_E(H_\Lambda))$$

which is known as Shubin's formula ([3], Section 2.7). When E lies in a spectral gap, $\chi_E(H_\Lambda)$ lies in the C^*-algebra $\mathcal{A}_\theta$. In this case, the value of the IDOS is constant over the gap and can be described using the trace on $\mathcal{A}_\theta$.

Thus there is physical interest in computing the image of the trace map

$$Tr_* : K_0(\mathcal{A}_\theta) \to \mathbb{R}$$