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Spectral triples for subshifts

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ABSTRACT

We propose a construction for spectral triple on algebras associated with subshifts. One-dimensional subshifts provide concrete examples of $\mathbb{Z}$-actions on Cantor sets. The C^*-algebra of this dynamical system is generated by functions in $C(X)$ and a unitary element u implementing the action. Building on ideas of Christensen and Ivan, we give a construction of a family of spectral triples on the commutative algebra $C(X)$. There is a canonical choice of eigenvalues for the Dirac operator D which ensures that $[D, u]$ is bounded, so that it extends to a spectral triple on the crossed product.

We study the summability of this spectral triple, and provide examples for which the Connes distance associated with it on the commutative algebra is unbounded, and some for which it is bounded. We conjecture that our results on the Connes distance extend to the spectral triple defined on the noncommutative algebra.

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1. Introduction

Alain Connes introduced the notion of a spectral triple which consists of a separable, infinite dimensional complex Hilbert space $\mathcal{H}$, a $*$-algebra of bounded linear operators on $\mathcal{H}$, A, and a self-adjoint unbounded operator D such that $(1 + D^2)^{-1}$ is a compact operator. The key relation is that for all a in A, $[D, a] = Da - aD$ is densely defined and extends to a bounded operator [9]. The motivating example is where A consists of the smooth functions on a compact manifold and D is some type of elliptic differential operator.

In spite of its geometric origins, there has been considerable interest in finding examples where the algebra A is the continuous functions on a compact, totally disconnected metric space with no isolated points. We refer to such a space as a Cantor set. The first example was given by Connes [8–10] but many other authors have also contributed: Sharp [27], Pearson–Bellissard [23], Christensen–Ivan [7], etc. Many of these results concern not just a Cantor set, but also some type of dynamical system on a Cantor set. Many such systems are closely related to aperiodic tilings and are important as mathematical models for quasicrystals.

We continue these investigations here by studying subshifts. Let $\mathcal{A}$ be a finite set (called the alphabet). Consider $\mathcal{A}^{\mathbb{Z}}$ with the product topology and the homeomorphism σ which is the left shift: $\sigma(x)_n = x_{n+1}, n \in \mathbb{Z}$, for all x in $\mathcal{A}^{\mathbb{Z}}$. A subshift X is any non-empty, closed and shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$. It is regarded as a dynamical system for the map $\sigma|_X$.

Our aim here is to construct and study examples of spectral triples for the algebra of locally constant functions on X, which we denote $C_\infty(X)$. Besides, we let $C(X)$ denote the C^*-algebra of continuous functions on X and $C(X) \times \mathbb{Z}$ be the crossed product C^*-algebra generated by $C(X)$ and a canonical unitary u. We will also consider the $*$-subalgebra of the crossed product generated by $C_\infty(X)$ and u and study how our spectral triples extend to this algebra.

Our construction is just a special case of that given by Christensen and Ivan for AF-algebras [7]. Given an increasing sequence of finite-dimensional C^*-algebras

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

one considers their union and a representation constructed by the GNS method. The same sequence of sets provides an increasing sequence of finite-dimensional subspaces for the associated Hilbert spaces which is used the construct the eigenspaces for the operator D. The one other piece of data needed for our construction is a σ-invariant measure with support X, which we denote μ.

Our case of the subshift has some special features. The first is that the sequence of finite-dimensional subalgebras we construct is canonical, arising from the structure as a subshift. Secondly, the choice of eigenvalues for the operator D is constrained by the