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Small analytic solutions to the Hartree equation

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ABSTRACT

We consider the Cauchy problem for the Hartree equation in space dimension $d \geq 3$. We assume that the interaction potential V belongs to the weak $L^{d/2}$ space. We prove that if the initial data ϕ is sufficiently small in the L^2-sense and either ϕ or its Fourier transform $\mathcal{F}\phi$ satisfies a real-analytic condition, then the solution $u(t)$ is also real-analytic for any $t \neq 0$. We also prove that if ϕ and V satisfy some strong condition, then $u(t)$ can be extended to an entire function on $\mathbb{C}^d$ for any $t \neq 0$. A part of our method is applicable to the final value problem. We remark that no L^2 smallness condition is imposed on first and higher order partial derivatives of ϕ and $\mathcal{F}\phi$.

1. Introduction

We consider the Cauchy problem for the nonlinear Schrödinger equation of the form

$$\begin{cases} iu_t + \Delta u = F(u), \\ u(0,x) = \phi(x). \end{cases} \tag{1.1}$$

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Here, u is a complex-valued unknown function of $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, $d \geq 3$, $i = \sqrt{-1}$, Δ is the Laplacian in $\mathbb{R}^d$, $F(u)$ denotes the Hartree term $(V * |u|^2)u$ and $*$ is the convolution in $\mathbb{R}^d$. Throughout this paper, we assume that the interaction potential V is a given function on $\mathbb{R}^d$ and belongs to the weak $L^{d/2}$ space. In other words, we assume that

$$\sup_{\lambda>0} \lambda\, \mu\Big(\big\{x \in \mathbb{R}^d; |V(x)| > \lambda\big\} \Big)^{2/d} < \infty, \tag{1.2}$$

where μ is the Lebesgue measure on $\mathbb{R}^d$. There is a large literature on the Cauchy problem for nonlinear Schrödinger equations (see, e.g., [2,13,25] and references therein).

To state a global existence theorem for (1.1), we set some notation. For $q \in [1,\infty]$, we denote the Lebesgue space $L^q(\mathbb{R}^d)$ and the L^q-norm by L^q and $\|\cdot\|_q$, respectively, and we set $\|\cdot\| = \|\cdot\|_2$. For $t \in \mathbb{R}$, $U(t)$ denotes the propagator $e^{it\Delta}$ for the free Schrödinger equation $iu_t + \Delta u = 0$. Mochizuki [14] has proved that if the condition

$$\text{either} \quad |V(x)| \leq C|x|^{-2} \quad \text{or} \quad V \in L^{d/2},$$

which is stronger than (1.2), holds and $\|\phi\|$ is sufficiently small, then there exists a time-global solution u to the integral equation of the form

$$u(t) = U(t)\phi - i\int_0^t U(t-t')F(u(t'))dt', \quad t \in \mathbb{R}, \tag{1.3}$$

which is equivalent to (1.1), such that $u(t)$ behaves like a free solution $U(t)\phi_+$ as $t \to \infty$ in L^2. In particular, the inverse wave operator $\mathbf{V}_+ : \phi \mapsto \phi_+$ is well-defined on a neighborhood of 0 in L^2 (see also [16]). Remark that in the above existence theorem, no L^2 smallness condition is imposed on first and higher order partial derivatives of ϕ and its Fourier transform $\mathcal{F}\phi$.

1.1. Main results

In this paper, assuming that either ϕ or $\mathcal{F}\phi$ satisfies a real-analytic condition, we study the analyticity of the small solution $u(t)$ to (1.3), the final data $\mathbf{V}_+(\phi)$ and $\mathcal{F}\mathbf{V}_+(\phi)$. Remark that we do not impose any L^2 smallness condition on any partial derivative of ϕ and $\mathcal{F}\phi$. We now briefly state a part of our main result. We show that:

(I) There exists some $\eta > 0$ such that if $0 < \|\phi\| < \eta$ and

$$\limsup_{|\alpha|\to\infty} \left(\frac{\|x^\alpha \phi\|}{\alpha!} \right)^{1/|\alpha|} < \infty, \tag{1.4}$$