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L^p-tauberian theorems and L^p-rates for energy decay ☆

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We prove L^p-analogues of the classical tauberian theorem of Ingham and Karamata, and its variations giving rates of decay. These results are applied to derive L^p-decay of operator families arising in the study of the decay of energy for damped wave equations and local energy for wave equations in exterior domains. By constructing some examples of critical behaviour we show that the L^p-rates of decay obtained in this way are best possible under our assumptions.

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1. Introduction

One of the basic results in tauberian theory is a theorem due to Ingham [23] and Karamata [24]. It has been used for elementary proofs of the prime number theorem, it has been a precursor for a number of famous results in function theory and operator theory such as theorems of Katznelson–Tzafriri type, and it still provides a link between many different applications of these theories. The result has found its way into many books and papers and has become a classic of modern tauberian theory; for a detailed discussion, see [28].

One version of the theorem of Ingham and Karamata reads as follows [5, Theorem 4.4.1], [28, Theorem III.7.1].

Theorem 1.1. *Let X be a Banach space and let $f \in L^\infty(\mathbb{R}_+, X)$ be such that the Laplace transform $\widehat{f}$ admits an analytic extension to each point of $i\mathbb{R}$. Then the improper integral $\int_0^\infty f(s)\,ds$ exists and equals $\widehat{f}(0)$.*

Theorem 1.1 can be equipped with rates as the next result shows. For a continuous increasing function $M : \mathbb{R}_+ \to [2, \infty)$, define

$$M_{\log}(s) := M(s)[\log(1 + M(s)) + \log(1 + s)], \tag{1.1}$$

$$\Omega_M := \left\{ \lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\frac{1}{M(|\operatorname{Im}\lambda|)} \right\}. \tag{1.2}$$

The function $M_{\log}$ is continuous, strictly increasing, with $\lim_{s\to\infty} M_{\log}(s) = \infty$, so it has an inverse function $M_{\log}^{-1}$ defined on $[a, \infty)$ for some $a > 0$. The following was established in [9] (see [5, Theorem 4.4.6]).

Theorem 1.2. *Let $f \in L^\infty(\mathbb{R}_+, X)$, and assume that $\widehat{f}$ extends analytically to Ω_M and the extension satisfies*

$$\|\widehat{f}(\lambda)\| \le M(|\operatorname{Im}\lambda|), \qquad \lambda \in \Omega_M. \tag{1.3}$$

Let $M_{\log}$ be defined as above, and $c \in (0, 1)$. Then there exist positive numbers C and t_0, depending only on $\|f\|_\infty$, M and c, such that

$$\left\| \widehat{f}(0) - \int_0^t f(s)\,ds \right\| \le \frac{C}{M_{\log}^{-1}(ct)}, \qquad t \ge t_0. \tag{1.4}$$

There are versions of the Ingham–Karamata theorem allowing for a “small” set of singularities of $\widehat{f}$ on the imaginary axis [4]. The following is the simple case (for example, see [5, Theorem 4.4.8]).