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Factoriality of Hecke–von Neumann algebras of right-angled Coxeter groups ☆

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ABSTRACT

The Hecke algebra $\mathbb{C}_q[W]$ of a Coxeter group W, associated to parameter q, can be completed to a von Neumann algebra $\mathcal{N}_q(W)$. We study such algebras in case where W is right-angled. We determine the range of q for which $\mathcal{N}_q(W)$ is a factor, i.e. has trivial center. Moreover, in case of nontrivial center, we prove a result allowing to decompose $\mathcal{N}_q(W)$ into a finite direct sum of factors.

1. Introduction

For $q > 0$ the Hecke algebra $\mathbb{C}_q[W]$ of a Coxeter group W is a deformation of the group algebra of W, consisting of finitely supported functions on W with a modified product, yielding the ordinary group algebra for $q = 1$. In case where q is an integer, the

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Hecke algebra has a nice geometric interpretation. Recall that a building of type W can be thought of as a space endowed with a W-valued metric [1, Chapter 5]. If it is locally finite, i.e. for every w there are finitely many points y at distance w from a fixed point x, then for every $w \in W$ we may consider an operator A_w on the space of finitely supported functions on the building, which averages the functions over spheres of radius w. If all spheres of radius s, where s is any of the standard generators of W, have cardinality q, the operators A_w generate an algebra isomorphic to $\mathbb{C}_q[W]$ [9]. For $q = 1$ the averages are taken over 1-element sets, and are in fact translations, so one indeed gets the group algebra.

The Hecke algebra has a natural action on the Hilbert space of square-integrable functions on W, obtained by extending its action on itself by left multiplication. It therefore completes to a von Neumann algebra—the Hecke–von Neumann algebra $\mathcal{N}_q(W)$. Its importance stems from the theory of weighted L^2-cohomology of Coxeter groups [6,4], where the cohomology spaces are modules over $\mathcal{N}_q(W)$.

A generalization of the Singer Conjecture [4, Conjecture 14.7] deals with vanishing of certain weighted cohomology spaces of W. The motivation for this work was to try to approach this problem using the central decomposition of $\mathcal{N}_q(W)$ to better understand the structure of these spaces. As it turns out, although the centers of $\mathcal{N}_q(W)$ can be nontrivial, they contribute nothing new in the subject of decomposing the weighted cohomology of W; the decompositions described in [4, Theorem 11.1] are finer than those induced by the central decomposition of $\mathcal{N}_q(W)$.

Although our results are not fit for the applications we initially had in mind, they are interesting in their own right. Namely, we show that the Hecke–von Neumann algebras of irreducible right-angled Coxeter groups are factors, up to a 1-dimensional direct summand. The main result of the paper reads as follows.

Main Theorem. *(See Theorem 5.3.) Suppose that (W, S) is an irreducible right-angled Coxeter system with W neither cyclic of order 2 nor infinite dihedral. Then the Hecke–von Neumann algebra $\mathcal{N}_q(W)$ is a factor if and only if*

$$q \in [\rho, \rho^{-1}], \tag{1.1}$$

where ρ is the convergence radius of $W(t)$, the growth series of W. Moreover, for q outside this interval, $\mathcal{N}_q(W)$ is a direct sum of a factor and $\mathbb{C}$.

Since taking centers commutes with tensor products, and the Hecke–von Neumann algebra $\mathcal{N}_q(W)$ of an arbitrary right-angled Coxeter group W is the tensor product of the Hecke–von Neumann algebras of irreducible factors of W, the central decomposition of $\mathcal{N}_q(W)$ follows in the general case.

The paper is divided into two parts. The first part consists of Sections 2 and 3, and is purely group-theoretic. Section 2 introduces the basic notions related to Coxeter groups, and in Section 3 we analyze certain double cosets in a right-angled Coxeter group, and use them to define the graph $\Gamma(W, S)$, whose connectivity is related to restrictions satisfied