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Lyapunov-type inequalities for partial differential equations

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ABSTRACT

In this work we present a Lyapunov inequality for linear and quasilinear elliptic differential operators in N-dimensional domains Ω. We also consider singular and degenerate elliptic problems with A_p coefficients involving the p-Laplace operator with zero Dirichlet boundary condition.
As an application of the inequalities obtained, we derive lower bounds for the first eigenvalue of the p-Laplacian, and compare them with the usual ones in the literature.

1. Introduction

In his classical work [23], Lyapunov proved that, given a continuous periodic and positive function w with period L, the solution u of the ordinary differential equation $u'' + w(t)u = 0$, in $(-\infty, +\infty)$, was stable if

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$$L \int_0^L w(t)dt < 4.$$

Then, Borg in [4] introduced the Lyapunov inequality in his proof of the stability criteria for sign changing weights w. He showed that the inequality

$$\frac{4}{L} \leq \int_0^L |w(t)|dt \tag{1.1}$$

must be satisfied in order to have a nontrivial solution in $[0, L] \subset \mathbb{R}$ of the problem

$$\begin{cases} u'' + w(t)u = 0, \\ u(0) = 0 = u(L). \end{cases} \tag{1.2}$$

Since then, it was rediscovered and generalized many times. Inequality (1.1) was applied in stability problems, oscillation theory, a priori estimates, other inequalities, and eigenvalue bounds for ordinary differential equations. Different proofs of this inequality have appeared in the literature: the proof of Patula [28] by direct integration, or the one of Nehari [24] showing the relationship with Green's functions, among several others. See the survey [5] for other proofs.

In the nonlinear setting, the following inequality

$$\frac{2^p}{L^{p-1}} \leq \int_0^L w(t)dt \tag{1.3}$$

generalized Lyapunov inequality (1.1) to p-Laplacian problems,

$$\begin{cases} (|u'|^{p-2}u')' + w(t)|u|^{p-2}u = 0, \\ u(0) = 0 = u(L). \end{cases}$$

Here, $w \in L^1$ and $1 < p < \infty$, for $p = 2$ we recover the linear problem (1.2). Several proofs were given in the last years, see [21,27,29,33]; although it seems to be derived first by Elbert [14].

Later, we extended it in [10] to nonlinear operators in Orlicz spaces generalizing the p-Laplacian,

$$-(\varphi(u'))' = \lambda r(t)\varphi(u), \tag{1.4}$$

where $\varphi(s)$ is a convex nondecreasing function, such that $s\varphi(s)$ satisfies the Δ_2 condition. Moreover, we also extend it to systems of resonant type (see [3]) involving p- and