Journal of Functional Analysis 270 (2016) 2019–2047

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Triple variational principles for self-adjoint operator functions

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ARTICLE INFO

Article history:
Received 13 June 2013
Accepted 3 September 2015
Available online 19 January 2016
Communicated by L. Gross

MSC:
primary 49R05
secondary 47A56, 47A10

Keywords:
Variational principles for eigenvalues
Operator functions
Spectral decomposition

ABSTRACT

For a very general class of unbounded self-adjoint operator functions we prove upper bounds for eigenvalues which lie within arbitrary gaps of the essential spectrum. These upper bounds are given by triple variations. Furthermore, we find conditions which imply that a point is in the resolvent set. For norm resolvent continuous operator functions we show that the variational inequality becomes an equality.

1. Introduction

In many applications of operator and spectral theory eigenvalue problems appear which are nonlinear in the eigenvalue parameter, e.g. polynomially or rationally. Very often such problems can be dealt with by introducing a function of the spectral parameter

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http://dx.doi.org/10.1016/j.jfa.2015.09.004

whose values are linear operators in a Hilbert space. To be more specific, let $T(\cdot)$ be an operator function that is defined on some set $\Delta \subset \mathbb{C}$ and whose values are closed linear operators in a Hilbert space $(\mathcal{H}, \langle \cdot\,, \cdot \rangle)$; for each $\lambda \in \Delta$ the domain of the operator $T(\lambda)$ is denoted by $\mathrm{dom}(T(\lambda))$. A number $\lambda \in \Delta$ is called an *eigenvalue* of the operator function T if there exists an $x \in \mathrm{dom}(T(\lambda)) \setminus \{0\}$ such that $T(\lambda)x = 0$, i.e. 0 is in the point spectrum of the operator $T(\lambda)$. The *spectrum*, *essential spectrum*, *discrete spectrum* and *resolvent set* of T are defined as follows:

$$\begin{aligned}
\sigma(T) &:= \bigl\{\lambda \in \Delta : 0 \in \sigma(T(\lambda))\bigr\},\\
\sigma_{\mathrm{ess}}(T) &:= \bigl\{\lambda \in \Delta : 0 \in \sigma_{\mathrm{ess}}(T(\lambda))\bigr\} = \bigl\{\lambda \in \Delta : T(\lambda) \text{ is not Fredholm}\bigr\},\\
\sigma_{\mathrm{dis}}(T) &:= \sigma(T) \setminus \sigma_{\mathrm{ess}}(T),\\
\rho(T) &:= \bigl\{\lambda \in \Delta : 0 \in \rho(T(\lambda))\bigr\};
\end{aligned}$$

note that a closed operator is called Fredholm if the dimension of the kernel and the (algebraic) co-dimension of the range are finite. A trivial example of an operator function is given by $T(\lambda) = A - \lambda I$ where A is a closed operator; in this case the spectra of the operator function T and the operator A clearly coincide. More complicated examples are operator polynomials or Schur complements of block operator matrices; see, e.g. [25,31] and the references therein; see also the survey article [29] about numerical methods for eigenvalues of quadratic matrix polynomials.

It is our aim to show spectral enclosures and variational principles for eigenvalues of operator functions. In the 1950s R.J. Duffin [6] proved a variational principle for eigenvalues of certain quadratic matrix polynomials, which was generalised to infinite-dimensional spaces and more general operator functions in the following decades; see, e.g. [27,32,14,36,1,25]. Basically, the following situation was considered. Let T be a differentiable function defined on an interval $[\alpha, \beta]$ whose values are bounded self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $T(\alpha) \gg 0$ (i.e. $T(\alpha)$ is uniformly positive) and $T(\beta) \ll 0$. Moreover, for every $x \in \mathcal{H}\setminus\{0\}$ the scalar function $\lambda \mapsto \langle T(\lambda)x, x\rangle$ has exactly one zero in (α, β), which we denote by $p(x)$, and the inequality $\langle T'(p(x))x, x\rangle < 0$ holds. The mapping $x \mapsto p(x)$ is called a generalised Rayleigh functional. The eigenvalues of T below the essential spectrum of T can accumulate at most at the bottom of $\sigma_{\mathrm{ess}}(T)$; if they are denoted by $\lambda_1 \le \lambda_2 \le \cdots$, then they are characterised by the following variational principle:

$$\lambda_n = \min_{\substack{\mathcal{L}\subset\mathcal{H}\\ \dim\mathcal{L}=n}} \; \max_{\substack{x\in\mathcal{L}\\ x\neq 0}} p(x) = \max_{\substack{\mathcal{L}\subset\mathcal{H}\\ \dim\mathcal{L}=n-1}} \; \min_{\substack{x\in\mathcal{H}\\ x\perp\mathcal{L},\ x\neq 0}} p(x); \tag{1.1}$$

here $\mathcal{L}$ denotes finite-dimensional subspaces of $\mathcal{H}$. If $T(\lambda) = A - \lambda I$ where A is a bounded self-adjoint operator, then (1.1) reduces to the standard variational principle for eigenvalues of a self-adjoint operator; the generalised Rayleigh functional is then just the classical Rayleigh quotient: $p(x) = \frac{\langle Ax, x\rangle}{\|x\|^2}$.