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Functions of unitary operators: Derivatives and trace formulas

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ABSTRACT

Let A be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and U_0 an arbitrary unitary operator. We derive formulas for higher order derivatives of the function $s \to \varphi(e^{isA}U_0)$ for sufficiently smooth functions φ. We compare our formulas with earlier results in this area and apply them to establish higher order analogs of trace formulas of Krein–Koplienko–Neidhardt.

1. Introduction

Let U_0 be a unitary and A a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Denote $U(s) = e^{\mathrm{i}sA}U_0$. In this paper we compute higher order Gâteaux derivatives of the operator function $s \to \varphi(U(s))$, where $\varphi : \mathbb{T} \to \mathbb{C}$ is a sufficiently smooth

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function, $\mathbb{C}$ is the field of all complex numbers, and $\mathbb{T}$ is the unit circle centered at 0. Our main results (Theorems 3.1 and 4.1) yield new formulas for derivatives $\frac{d^n}{ds^n}\big|_{s=t}\varphi(U(s))$ and establish existence of spectral shift functions of higher order for the couple of unitaries U_0 and $U_1 = e^{iA}U_0$.

More precisely, for an arbitrary function φ from the Besov class $B^n_{\infty,1}$, $n \in \mathbb{N}$, we realize the value $\frac{d^n}{ds^n}\big|_{s=t}\varphi(U(s))$ as a linear combination of multiple operator integrals defined with respect to divided differences $\varphi^{[1]}, \varphi^{[2]}, \dots, \varphi^{[n]}$ of the function φ (see Theorem 3.1). The case $n = 1$ (respectively, $n \geq 2$) was treated in [4,12] (respectively [13]). However, the proof in [13] contains some mistakes that lead to an incorrect formula there. These mistakes are avoided in the present approach. Alternatively, one could derive the aforementioned formulas for $\frac{d^n}{ds^n}\big|_{s=t}\varphi(U(s))$ by following the approach of [1, Section 5].

In Theorem 4.1, we apply the results of Theorem 3.1 and [15] to establish representations for the remainders of the Taylor approximations

$$tr\left(\varphi(e^{iA}U_0) - \varphi(U_0) - \sum_{k=1}^{n-1} \frac{1}{k!}\frac{d^k}{ds^k}\Big|_{s=0}\varphi(e^{isA}U_0)\right) = \int_0^{2\pi} \varphi^{(n)}(e^{it})\eta_n(t)\,dt \tag{1.1}$$

analogous to the well known trace formulas [9,8,14,2]

$$tr\left(f(H_0+V) - f(H_0) - \sum_{k=1}^{n-1} \frac{1}{k!}\frac{d^k}{dt^k}\Big|_{t=0} f(H_0+tV)\right) = \int_{-\infty}^{\infty} f^{(n)}(t)\eta_{n,sa}(t)\,dt \tag{1.2}$$

where H_0 and V are self-adjoint operators and $n \in \mathbb{N}$. Here η_n and $\eta_{n,sa}$ are integrable functions, called spectral shift functions, and φ and f are sufficiently nice scalar functions. Trace formulas for pairs of contractions with operator derivatives evaluated as in (1.2) were established in [15]. (Further references on the trace formulas can be found in, e.g., [19].) The formula (1.1) in case $n = 1$ was established in [10]. Another trace formula in case $n = 2$ was established in [11], but it is quite different from (1.1). This is discussed in more detail in Remark 4.7.

2. Preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$ and let Tr be the standard trace on $\mathcal{B}(\mathcal{H})$. By $\mathcal{S}^\alpha$ $(1 \leq \alpha < \infty)$ we denote a Schatten–von Neumann ideal of compact operators with its standard norm $\|\cdot\|_\alpha$ (see e.g. [18]). We shall also use the notation $\mathcal{S}^\infty$ as a shorthand for $\mathcal{B}(\mathcal{H})$ and $\|\cdot\|$ for a uniform operator norm on $\mathcal{B}(\mathcal{H})$.

We use standard notation $L^1 = L^1(\mathbb{T})$ and $L^\infty = L^\infty(\mathbb{T})$ for the Banach spaces of all Lebesgue integrable and essentially bounded functions on $\mathbb{T}$, respectively, and $H^p = H^p(\mathbb{T})$ for the Hardy space defined by $\{f \in L^p \ : \ \hat{f}(n) = 0 \text{ for all } n < 0\}$, $1 \leq p \leq \infty$. We also denote by $C(\mathbb{T})$ the Banach space of all continuous functions on $\mathbb{T}$ with the standard norm and by $\mathcal{G}_n(\mathbb{T})$ and $\mathcal{G}_{0,n}(\mathbb{T})$, $n \in \mathbb{N}$, we denote the linear subspaces of $C(\mathbb{T})$ defined by