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Estimating quantum chromatic numbers ☆

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We develop further the new versions of quantum chromatic numbers of graphs introduced by the first and fourth authors. We prove that the problem of computation of the commuting quantum chromatic number of a graph is solvable by an SDP algorithm and describe an hierarchy of variants of the commuting quantum chromatic number which converge to it. We introduce the tracial rank of a graph, a parameter that gives a lower bound for the commuting quantum chromatic number and parallels the projective rank, and prove that it is multiplicative. We describe the tracial rank, the projective rank and the fractional chromatic numbers in a unified manner that clarifies their connection with the commuting quantum chromatic number, the quantum chromatic number and the classical chromatic number, respectively. Finally, we present a new SDP algorithm that yields a parameter larger

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than the Lovász number and is yet a lower bound for the tracial rank of the graph. We determine the precise value of the tracial rank of an odd cycle.

1. Introduction

We assume that the reader is familiar with some concepts from graph theory and refer the reader to the text [6] for any terminology that we do not explain.

In [5,1,2,15] the concept of the quantum chromatic number $\chi_{\mathrm{q}}(G)$ of a graph G was developed and inequalities for estimating this parameter, as well as methods for its computation, were presented. In [12] several new variants of the quantum chromatic number, denoted $\chi_{\mathrm{qc}}(G)$, $\chi_{\mathrm{qa}}(G)$ and $\chi_{\mathrm{qs}}(G)$, were introduced, as well as $\chi_{\mathrm{vect}}(G)$. The motivation behind them came from conjectures of Tsirelson and Connes and the fact that the set of correlations of quantum experiments may possibly depend on which set of quantum mechanical axioms one chooses to employ. Given a graph G, the aforementioned chromatic numbers satisfy the inequalities

$$\chi_{\mathrm{vect}}(G) \leq \chi_{\mathrm{qc}}(G) \leq \chi_{\mathrm{qa}}(G) \leq \chi_{\mathrm{qs}}(G) \leq \chi_{\mathrm{q}}(G) \leq \chi(G),$$

where $\chi(G)$ denotes the classical chromatic number of the graph G.

The motivation of [12] for defining and studying these new chromatic numbers comes from the fact that if Tsirelson's conjecture is true, then $\chi_{\mathrm{qc}}(G) = \chi_{\mathrm{q}}(G)$ for every graph G, while if Connes' Embedding Problem has an affirmative answer, then $\chi_{\mathrm{qc}}(G) = \chi_{\mathrm{qa}}(G)$ for every graph G. Thus, computing these invariants gives a means to test the corresponding conjectures.

In [3] it was shown that

$$\lceil \vartheta^{+}(\overline{G}) \rceil = \chi_{\mathrm{vect}}(G),$$

where $\lceil r \rceil$ denotes the least integer greater than or equal to r and ϑ^{+} is Szegedy's [17] variant of Lovász's [9] ϑ-function. Furthermore, this identity was used to give the first example of a graph for which $\chi_{\mathrm{vect}}(G) \neq \chi_{\mathrm{q}}(G)$. Also, since ϑ^{+} is defined by an SDP, the aforementioned result shows that $\chi_{\mathrm{vect}}(G)$ is computable by an SDP.

In this paper we show that for each size of graph, $\chi_{\mathrm{qc}}(G)$ is also computable by an SDP. Our proof builds on ideas borrowed from the "NPA hierarchy" exhibited in [10]. It uses a compactness argument to show that for the purposes of computing this integer the hierarchy terminates, but does not yield the stage at which it does so. Thus, while we can say that it is computable by one of the SDP's in the hierarchy, we cannot explicitly determine the size of this SDP. It is known that $\chi(G)$ is computable by an SDP, but it is still not known if the same is true for $\chi_{\mathrm{qa}}(G)$, $\chi_{\mathrm{qs}}(G)$ and $\chi_{\mathrm{q}}(G)$. If the Tsirelson and