Journal of Functional Analysis 270 (2016) 3000–3046

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Nevanlinna representations in several variables ☆

J. Agler [a], R. Tully-Doyle [b], N.J. Young [c,d,*]

[a] *Department of Mathematics, University of California at San Diego, CA 92103, USA*
[b] *Department of Mathematics, Hampton University, Hampton, VA 23668, USA*
[c] *School of Mathematics, Leeds University, Leeds LS2 9JT, England, United Kingdom*
[d] *School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE3 4LR, England, United Kingdom*

ARTICLE INFO

Article history:
Received 28 December 2014
Accepted 2 February 2016
Available online 23 February 2016
Communicated by G. Schechtman

MSC:
32A30
30E20
30E05
47B25
47A10

Keywords:
Pick class
Cauchy transform
Self-adjoint operator
Resolvent

ABSTRACT

We generalize to several variables the classical theorem of Nevanlinna that characterizes the Cauchy transforms of positive measures on the real line. We show that for the Loewner class, a large class of analytic functions that have non-negative imaginary part on the upper polyhalf-plane, there are representation formulae in terms of densely-defined self-adjoint operators on a Hilbert space. We find four different representation formulae and we show that every function in the Loewner class has one of the four representations, corresponding precisely to four different growth conditions at infinity.

☆ The first author was partially supported by the National Science Foundation Grant on Extending Hilbert Space Operators DMS 1068830 and 1361720. The third author was partially supported by the UK Engineering and Physical Sciences Research Council grants EP/J004545/1 and EP/N03242X/1.

* Corresponding author at: School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE3 4LR, England, United Kingdom.

E-mail addresses: ryan.tullydoyle@hamptonu.edu (R. Tully-Doyle), Nicholas.Young@ncl.ac.uk (N.J. Young).

http://dx.doi.org/10.1016/j.jfa.2016.02.004

1. Introduction

In a classic paper [18] of 1922 R. Nevanlinna solved the problem of the determinacy of solutions of the Stieltjes moment problem. *En route* he proved several other theorems that have since been influential; in particular, the following theorem, which characterizes the Cauchy transforms of positive finite measures μ on $\mathbb{R}$, has had a profound impact on the development of modern analysis. Let $\mathcal{P}$ denote the Pick class, that is, the set of analytic functions on the upper half-plane,

$$\Pi \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z > 0\},$$

that have non-negative imaginary part on Π.

Theorem 1.1 *(Nevanlinna's Representation). Let h be a function defined on Π. There exists a finite positive measure μ on $\mathbb{R}$ such that*

$$h(z) = \int \frac{\mathrm{d}\mu}{t - z} \tag{1.1}$$

if and only if $h \in \mathcal{P}$ and

$$\liminf_{y \to \infty} y\, |h(iy)| < \infty. \tag{1.2}$$

A closely related theorem, also referred to in the literature as Nevanlinna's Representation, provides an integral representation for a general element of $\mathcal{P}$.

Theorem 1.2. *A function $h : \Pi \to \mathbb{C}$ belongs to the Pick class $\mathcal{P}$ if and only if there exist $a \in \mathbb{R}$, $b \geq 0$ and a finite positive Borel measure μ on $\mathbb{R}$ such that*

$$h(z) = a + bz + \int \frac{1 + tz}{t - z}\, \mathrm{d}\mu(t) \tag{1.3}$$

for all $z \in \Pi$. Moreover, for any $h \in \mathcal{P}$, the numbers $a \in \mathbb{R}$, $b \geq 0$ and the measure $\mu \geq 0$ in the representation (1.3) are uniquely determined.

What are the several-variable analogs of Nevanlinna's theorems? In this paper we shall propose four types of Nevanlinna representation for various subclasses of the n-variable Pick class $\mathcal{P}_n$, where $\mathcal{P}_n$ is defined to be the set of analytic functions h on the polyhalf-plane Π^n such that $\operatorname{Im} h \geq 0$. In addition, we shall present necessary and sufficient conditions for a function defined on Π^n to possess a representation of a given type in terms of asymptotic growth conditions at ∞.

The integral representation (1.1) of those functions in the Pick class that satisfy condition (1.2) can be written in the form