Journal of Functional Analysis 270 (2016) 2483–2507

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Diffusion with nonlocal boundary conditions ☆

Wolfgang Arendt [a,*], Stefan Kunkel [b], Markus Kunze [b]

[a] *Institute of Applied Analysis, Ulm University, 89069 Ulm, Germany*
[b] *Graduiertenkolleg 1100, Ulm University, 89069 Ulm, Germany*

A R T I C L E   I N F O

Article history:
Received 20 September 2014
Accepted 28 January 2016
Available online 15 February 2016
Communicated by F. Otto

MSC:
47D07
60J35
35B35

Keywords:
Diffusion process
Nonlocal boundary conditions
Asymptotic behavior

A B S T R A C T

We consider second order differential operators A_μ on a bounded, Dirichlet regular set $\Omega \subset \mathbb{R}^d$, subject to the nonlocal boundary conditions

$$u(z) = \int_\Omega u(x)\, \mu(z, dx) \quad \text{for } z \in \partial\Omega.$$

Here the function $\mu : \partial\Omega \to \mathscr{M}^+(\Omega)$ is $\sigma(\mathscr{M}(\Omega), C_b(\Omega))$-continuous with $0 \le \mu(z, \Omega) \le 1$ for all $z \in \partial\Omega$. Under suitable assumptions on the coefficients in A_μ, we prove that A_μ generates a holomorphic positive contraction semigroup T_μ on $L^\infty(\Omega)$. The semigroup T_μ is never strongly continuous, but it enjoys the strong Feller property in the sense that it consists of kernel operators and takes values in $C(\overline{\Omega})$. We also prove that T_μ is immediately compact and study the asymptotic behavior of $T_\mu(t)$ as $t \to \infty$.

☆ S.K. and M.K. were supported by the *Deutsche Forschungsgemeinschaft* in the framework of the DFG research training group 1100.

* Corresponding author.
E-mail addresses: wolfgang.arendt@uni-ulm.de (W. Arendt), stefan.kunkel@uni-ulm.de (S. Kunkel), markus.kunze@uni-ulm.de (M. Kunze).

http://dx.doi.org/10.1016/j.jfa.2016.01.025

1. Introduction

W. Feller [13,14] has studied one-dimensional diffusion processes and their corresponding transition semigroups. In particular, he characterized the boundary conditions which must be satisfied by functions in the domain of the generator of the transition semigroup. These include besides the classical Dirichlet and Neumann boundary conditions also certain nonlocal boundary conditions.

Subsequently, A. Venttsel' [31] considered the corresponding problem for diffusion problems on a domain $\Omega \subset \mathbb{R}^d$ with smooth boundary. He characterized boundary conditions which can potentially occur for the generator of a transition semigroup. Naturally, the converse question of proving that a second order elliptic operator (or more generally, an integro-differential operator) subject to certain (nonlocal) boundary conditions indeed generates a transition semigroup, has received a lot of attention, see the article by Galakhov and Skubachevskiĭ [15], the book by Taira [30] and the references therein. We would like to point out that the interest in general – also nonlocal – boundary conditions is not only out of mathematical curiosity. In fact, nonlocal boundary conditions appear naturally in applications, e.g. in thermoelasticity [11], in climate control systems [20] and in financial mathematics [17].

In this article, we consider second order differential operators on a bounded open subset Ω of $\mathbb{R}^d$, formally given by

$$\mathscr{A}u := \sum_{ij=1}^{d} a_{ij}D_iD_ju + \sum_{j=1}^{d} b_jD_ju + c_0u.$$

Below, we will define a realization A_μ of $\mathscr{A}$ on $C(\overline{\Omega})$ subject to nonlocal boundary conditions of the form

$$u(z) = \int_\Omega u(x)\,\mu(z,dx)$$

for all $z \in \partial\Omega$, where $\mu : \partial\Omega \to \mathscr{M}^+(\Omega)$ is a measure-valued function with $0 \leq \mu(z,\Omega) \leq 1$. Here, $\mathscr{M}(\Omega)$ denotes the space of all (complex) Borel measures on Ω and $\mathscr{M}^+(\Omega)$ refers to the cone of positive measures.

This boundary condition has a clear probabilistic interpretation. Whenever a diffusing particle reaches the boundary (say at the point $z \in \partial\Omega$), it immediately jumps back to the interior Ω. The point it jumps to is chosen randomly, according to the distribution $\mu(z,\cdot)$. In the case where $\mu(z,\Omega) < 1$, the particle "dies" with probability $1 - \mu(z,\Omega)$. This is a multidimensional version of what Feller called in [14] an *instantaneous return process*. Stochastic processes of this form were constructed by Grigorescu and Kang [17,18] and Ben-Ari and Pinsky [6,7].

We will now make precise our assumptions on Ω, the coefficients a_{ij}, b_j and c_0 as well as the measures μ. Unexplained terminology will be discussed in Section 3.