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Uniqueness results for inverse Robin problems with bounded coefficient

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A B S T R A C T

In this paper we address the uniqueness issue in the classical Robin inverse problem on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, with L^∞ Robin coefficient, L^2 Neumann data and conductivity of class $W^{1,r}(\Omega)$, $r > n$. We show that uniqueness of the Robin coefficient on a subpart of the boundary, given Cauchy data on the complementary part, does hold in dimension $n = 2$ but needs not hold in higher dimension. We also raise on open issue on harmonic gradients which is of interest in this context.

1. Introduction

This study deals with uniqueness issues for the classical Robin inverse boundary value problem. Mathematically speaking, the inverse Robin problem for an elliptic partial differential equation on a domain consists in finding the ratio between the normal derivative and the trace of the solution (the so-called Robin coefficient) on a subset of the boundary, granted the Cauchy data (*i.e.* the normal derivative and the trace of the solution) on the

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complementary subset. In this paper, we deal primarily with L^∞ Robin coefficients and L^2 Neumann data, for isotropic conductivity equations of the type div $(\sigma \operatorname{grad} u) = 0$ on Lipschitz domains $\Omega \subset \mathbb{R}^n$, with Sobolev-smooth real-valued strictly elliptic conductivity σ of class $W^{1,r}(\Omega)$, $r > n$. An anisotropic analog to our uniqueness result is discussed in a separate section.

The Robin inverse problem arises for example when considering non-destructive testing of corrosion in an electrostatic conductor. In this case, data consist of surface measurements of both the current and the voltage on some (accessible) part of the boundary of the conductor, while the complementary (inaccessible) part of the boundary is subject to corrosion. Non-destructive testing consists in quantifying corrosion from the data. Robin boundary condition can be regarded as a simple model for corrosion [33]. Indeed, as was proved in [16], such boundary conditions arise when considering a thin oscillating coating surrounding a homogeneous background medium such that the thickness of the layer and the wavelength of the oscillations tend simultaneously to 0. A mathematical framework for corrosion detection can then be described as follows. We consider a conductivity equation in an open domain Ω, as a generalization of Laplace equation to non-homogeneous media, the boundary of which is divided into two parts. The first part Γ is characterized by a homogeneous Robin condition with functional coefficient λ. A non-vanishing flux is imposed on the second part Γ_0 of the boundary. This provides us with a well-posed forward problem, that is, there uniquely exists a solution in Ω meeting the prescribed boundary conditions. The inverse problem consists in recovering the unknown Robin coefficient λ on Γ from measurements of the trace of the solution on Γ_0. Further motivation to solve the Robin problem are indicated in [39] and its bibliography.

A basic question is uniqueness: is the coefficient λ on Γ uniquely defined by the available Cauchy data on Γ_0 as soon as the latter has positive measure? In other words, can we find two different Robin coefficients that produce the same measurements? The answer naturally depends on the smoothness assumed for λ.

On smooth domains, for the Laplace operator at least, uniqueness of the inverse Robin problem for (piecewise) continuous λ has been known for decades to hold in all dimensions. The proof is for example given in [33], and in [23] for the Helmholtz equation. It relies on a strong unique continuation property (Holmgren's theorem), *i.e.* on the fact that a harmonic function in Ω, the trace and normal derivative of which both vanish on a non-empty open subset of the boundary $\partial\Omega$, vanishes identically.

This argument no longer works for functions λ that are merely bounded. In this case we meet the following weaker unique continuation problem: *does a harmonic function, the trace and normal derivative of which both vanish on a subset of $\partial\Omega$ with positive measure, vanish identically?* A famous counterexample in [14] shows that such a unique continuation result is false in dimension 3 and higher. In dimension 2, a proof that such a unique continuation property holds for the Laplace equation can be found in [5] when the solution is assumed to be C^1 up to the boundary and Ω is the unit disk.