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Estimates of first and second order shape derivatives in nonsmooth multidimensional domains and applications

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In this paper we investigate continuity properties of first and second order shape derivatives of functionals depending on second order elliptic PDEs around nonsmooth domains, essentially either Lipschitz or convex, or satisfying a uniform exterior ball condition. We prove rather sharp continuity results for these shape derivatives with respect to Sobolev norms of the boundary-traces of the displacements. With respect to previous results of this kind, the approach is quite different and is valid in any dimension $N \geq 2$. It is based on sharp regularity results for Poisson-type equations in such nonsmooth domains. We also enlarge the class of functionals and PDEs for which these estimates apply. Applications are given to qualitative properties of shape optimization problems under convexity constraints for the variable domains or their complement.

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1. Introduction

In this paper, we focus on regularity estimates for first and second order shape derivatives *around nonsmooth subsets of* $\mathbb{R}^N$ *($N \geq 2$)* for energy functionals involving classical elliptic partial differential equations (PDEs). For instance, we address the following question: given a bounded Lipschitz or convex subset $\Omega_0 \subset \mathbb{R}^N$, we wonder what is the optimal regularity of the shape derivatives

$$\xi \to E'(\Omega_0)(\xi), \quad \xi \to E''(\Omega_0)(\xi, \xi),$$

where $E'(\Omega_0), E''(\Omega_0)$ respectively denote the first and second shape derivatives around Ω_0 of the shape functional $\Omega \mapsto E(\Omega) = \int_\Omega K(x, U_\Omega, \nabla U_\Omega)dx$, $K(x, \cdot, \cdot)$ a quadratic polynomial and $U_\Omega = U_\Omega(x)$ the solution of an elliptic PDE set in Ω (see Sections 2.1, 2.2 for the precise definitions).

This question, interesting in itself, is in particular motivated by the qualitative analysis of shape optimization problems of the form

$$\min\{J(\Omega),\ \Omega \subset \mathbb{R}^N \ \textbf{convex},\ \Omega \in \mathcal{O}_{ad}\}, \tag{1}$$

where $\mathcal{O}_{ad}$ is a set of admissible subsets of $\mathbb{R}^N$ and $J : \mathcal{O}_{ad} \to \mathbb{R}$ is a shape functional which itself involves shape functionals $\Omega \mapsto E(\Omega)$ of the above type.

The following 2-dimensional example was considered in [16] among other cases:

$$\begin{aligned} J(\Omega) &= R(E(\Omega), |\Omega|) - P(\Omega), \\ \mathcal{O}_{ad} &= \left\{\Omega \subset \mathbb{R}^2,\ \text{open and } \partial\Omega \subset \{x,\ a \leq |x| \leq b\}\right\}, \end{aligned} \tag{2}$$

where $R : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, $E(\Omega)$ is a shape functional related to a PDE, $|\Omega|$ is the Lebesgue measure of Ω, $P(\Omega)$ its perimeter and $(a, b) \in [0, \infty]^2$. It was proved (see [16, Theorem 2.9, Theorem 2.12 and Corollary 2.13] and also Section 4 in the present paper) that if the second order shape derivative $\xi \mapsto E''(\Omega_0)(\xi, \xi)$ around any bounded convex subset Ω_0 is *continuous with respect to a norm strictly weaker than* $H^1(\partial\Omega_0)$, then optimal shapes of (1) are *polygonal* in $\{x,\ a < |x| < b\}$. In [16], the authors prove that such a continuity holds in the two specific examples: when the functional $E(\Omega)$ is the Dirichlet energy of Ω – that is $K = K(x, U, q) = \|q\|^2 - 2f(x)$ and U_Ω is solution of the associated Dirichlet problem, or when $E(\Omega)$ is the first eigenvalue of the Dirichlet–Laplacian on Ω. More precisely, it is proved in [16] that the second order shape derivative of $E(\cdot)$ is, in these two examples and around any open convex domain Ω_0, continuous for the $H^{1/2}(\partial\Omega_0) \cap L^\infty(\partial\Omega_0)$ topology (and therefore continuous for the $H^{1/2+\varepsilon}(\partial\Omega_0)$-topology for any $\varepsilon > 0$). The proof of this continuity strongly relies on the 2-dimensional environment. As explained below, we prove in this paper that *even the full* $H^{1/2}(\partial\Omega_0)$*-continuity holds in this case and even in any dimension* (see iii) in Corollary 2.6). Note that this continuity is optimal since, for regular convex domains Ω_0, if for instance $f = 0$ on $\partial\Omega_0$, then $E''(\Omega_0)$ satisfies