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Banach lattice versions of strict singularity

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We explore the relation between lattice versions of strict singularity for operators from a Banach lattice to a Banach space. In particular, we study when the class of disjointly strictly singular operators, those not invertible on the span of any disjoint sequence, coincides with that of lattice strictly singular operators, i.e. those not invertible on any (infinite dimensional) sublattice. New results are given which help to clarify the existing relation between these two classes.

1. Introduction

Recall that a linear operator between Banach spaces $T : X \to Y$ is strictly singular (SS in short) if it is not an isomorphism when restricted to any infinite dimensional

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subspace. These operators form a two-sided operator ideal, and were introduced by T. Kato [12] in connection with the perturbation theory of Fredholm operators. More recently, this class has played an important role in the theory of hereditarily indecompossable spaces.

In this note we explore the relation between lattice versions of strict singularity. Given a Banach lattice E and a Banach space X, a bounded operator $T : E \to X$ is called *disjointly strictly singular* (DSS in short) if there is no infinite sequence $(x_n)_n$ of pairwise disjoint vectors in E such that the restriction of T to its closed linear span $[x_n]$ is an isomorphism. The class of DSS operators was introduced in [8] in connection with the structure of ℓ_p subspaces in Orlicz spaces. Recently, it proved to be a useful tool in the study of strictly singular operators on Banach lattices (cf. [4]).

Also, $T : E \to X$ is called *lattice strictly singular* (LSS in short) if for every infinite dimensional closed sublattice $S \subset E$ the restriction $T|_S$ is not an isomorphism. Notice that $T : E \to X$ is LSS if, and only if, there is no infinite sequence $(x_n)_n$ of *positive*, pairwise disjoint vectors in E such that the restriction of T to the span $[x_n]$ is an isomorphism. This follows from the fact that the linear space generated by a sequence of positive disjoint vectors is always a sublattice, and that every infinite dimensional sublattice contains a sequence of positive, disjoint vectors. Note that the linear span of a sequence of (non-positive) disjoint vectors is not a sublattice in general.

The classes of LSS and DSS operators contain all strictly singular operators. A natural example of a DSS (and LSS) operator which is not strictly singular is the formal identity $i : L_p(\mu) \to L_q(\mu)$ for $p > q$ and μ a finite measure: It is easy to see that this operator cannot be an isomorphism on any subspace isomorphic to ℓ_p, but due to Khintchine's inequality it is an isomorphism on a subspace isomorphic to ℓ_2.

Clearly, if an operator T is DSS then it is LSS. The main question is whether the converse holds:

Question 1. *Is every LSS operator also DSS?*

A motivation for this question can be traced back to the following observation in [7]: If an operator $T : E \to X$ is invertible in a subspace isomorphic to c_0 generated by a disjoint sequence, then T is also invertible in a sublattice isomorphic to c_0. In the current terminology, this is equivalent to saying that, given a compact Hausdorff space K and a Banach space X, every operator $T : C(K) \to X$ is LSS if and only if it is DSS.

A related question is whether the class of LSS operators forms a subspace of the space of bounded operators between a Banach lattice and a Banach space. More precisely,

Question 2. *Given two LSS operators $T_1, T_2 : E \to X$, is the sum $T_1 + T_2$ also LSS?*

Notice that the usual proof that shows that the sum of SS operators is SS still works in the class of DSS operators, but it does not in the class of LSS operators. Therefore, a positive answer to Question 1 would yield an affirmative answer to Question 2. Sur-