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# The best constants for operator Lipschitz functions on Schatten classes

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Suppose that $f$ is a Lipschitz function on $\mathbb{R}$ with $\|f\|_{\mathrm{Lip}} \leq 1$. Let $A$ be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $p \in (1, \infty)$ and suppose that $x \in B(\mathcal{H})$ is an operator such that the commutator $[A, x]$ is contained in the Schatten class $\mathcal{S}_p$. It is proved by the last two authors, that then also $[f(A), x] \in \mathcal{S}_p$ and there exists a constant $C_p$ independent of $x$ and $f$ such that

$$\big\|[f(A), x]\big\|_p \leq C_p \big\|[A, x]\big\|_p.$$

The main result of this paper is to give a sharp estimate for $C_p$ in terms of $p$. Namely, we show that $C_p \sim \frac{p^2}{p-1}$. In particular, this gives the best estimates for operator Lipschitz inequalities.

We treat this result in a more general setting. This involves commutators of $n$ self-adjoint operators $A_1, \ldots, A_n$, for which we prove the analogous result. The case described here in the abstract follows as a special case.

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## 1. Introduction

Recently, the last two authors proved that Lipschitz functions on $\mathbb{R}$ act as operator Lipschitz functions on the Schatten classes $\mathcal{S}_p$ for all $p \in (1,\infty)$, see [17,20]. That is, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and

$$\|f\|_{\mathrm{Lip}} = \sup_{\xi,\tilde{\xi}\in\mathbb{R}} \frac{|f(\xi) - f(\tilde{\xi})|}{\|\xi - \tilde{\xi}\|_1} \leq 1.$$

Let $p \in (1,\infty)$. Suppose that $A$, $B$ are bounded, self-adjoint operators such that $A - B \in \mathcal{S}_p$. Then, it was proved in [17] that also $f(A) - f(B) \in \mathcal{S}_p$ and there is a constant $C_p < \infty$ independent of $A$, $B$ and $f$ such that

$$\big\|f(A) - f(B)\big\|_p \leq C_p\|A - B\|_p. \tag{1.1}$$

We denote $C_p$ for the minimal constant for which the inequality (1.1) holds.

For the case $p = 1$, the analogous result fails. That is, there is no constant $C_1$ such that the inequality (1.1) holds as was proved in [5]. For the case $p = \infty$ the analogous statement also fails as was proved in [12].

This raises the question of what the growth order of $C_p$ is as $p$ approaches either 1 or $\infty$. In [13] it was proved that $C_p \preceq p^8$ as $p \to \infty$ and $C_p \preceq (p-1)^{-8}$ as $p \downarrow 1$. In fact, in [13] a more general result is covered involving an $n$-tuple of commuting self-adjoint (bounded) operators. We refer to [13, Theorem 5.3] for the precise statement.

In [17] an estimate for the asymptotic behaviour of $C_p$ was not mentioned explicitly. However, it is in principle possible to find an upper estimate for $C_p$ from the proof presented in [17]. These proofs involve the Marcinkiewicz multiplier theorem as well as diagonal truncation and do not lead to a sharp upper estimate of $C_p$.

The main result of this paper is a sharp estimate for $C_p$. Namely, we prove that $C_p \sim p$ as $p \to \infty$ and we prove that $C_p \sim (p-1)^{-1}$ as $p \downarrow 1$. Our result is stated in terms of commutator estimates in Schatten classes. In particular, it sharpens the estimates found in [13] for $n$-tuples of commuting self-adjoint operators.

The novelty of our proof is that we apply the main result of [6]. In [6] sharp estimates were found for the action of a smooth, even multiplier that acts on vector-valued $L^p$-spaces. The norm of such a multiplier can be expressed in terms of the UMD-constant of a Banach space (we recall the definition below). This result together with the so-called transference method forms the key argument that allows us to improve the known estimates for $C_p$.

This paper relates to the general interest of finding the best constants in non-commutative probability inequalities. In particular, major achievements have been made