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Boundedness of a class of bi-parameter square functions in the upper half-space

Henri Martikainen [1]

Département de Mathématiques, Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud 11, F-91405, Orsay Cedex, France

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ABSTRACT

We consider a class of bi-parameter kernels and related square functions in the upper half-space, and give an efficient proof of a boundedness criterion for them. The proof uses modern probabilistic averaging methods and is based on controlling double Whitney averages over good cubes.

1. Introduction

In this paper we introduce a class of bi-parameter kernels $(t_1, t_2, x_1, x_2, y_1, y_2) \mapsto s_{t_1,t_2}(x_1, x_2, y_1, y_2)$, where $t_1, t_2 > 0$ and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^{n+m}$. We then consider bi-parameter square functions

E-mail address: henri.martikainen@helsinki.fi.

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$$Sf(y_1, y_2) = \left(\iint\limits_{\Gamma(y_2)} \iint\limits_{\Gamma(y_1)} \left|\theta_{t_1,t_2} f(x_1, x_2)\right|^2 \frac{dx_1 dt_1}{t_1^{n+1}} \frac{dx_2 dt_2}{t_2^{m+1}} \right)^{1/2},$$

where $\Gamma(y_1) = \{(x_1, t_1) \in \mathbb{R}^{n+1}_+ \colon |x_1 - y_1| < t_1\}$ and

$$\theta_{t_1,t_2} f(x_1, x_2) = \iint\limits_{\mathbb{R}^{n+m}} s_{t_1,t_2}(x_1, x_2, z_1, z_2) f(z_1, z_2)\, dz_1\, dz_2.$$

The kernels s_{t_1,t_2} are assumed to satisfy a natural size estimate, a Hölder estimate and two symmetric mixed Hölder and size estimates. We also assume certain mixed Carleson and size estimates, mixed Carleson and Hölder estimates and a bi-parameter Carleson condition. Under these assumptions we show the square function bound

$$\|Sf\|^2_{L^2(\mathbb{R}^{n+m})} = C_n C_m \iint\limits_{\mathbb{R}^{m+1}_+} \iint\limits_{\mathbb{R}^{n+1}_+} \left|\theta_{t_1,t_2} f(x_1, x_2)\right|^2 \frac{dx_1 dt_1}{t_1} \frac{dx_2 dt_2}{t_2} \lesssim \|f\|^2_{L^2(\mathbb{R}^{n+m})}.$$

Compared to the bi-parameter Calderón–Zygmund theory the square function case is significantly cleaner. Indeed, the amount of needed symmetries and conditions are greatly reduced. Moreover, one encounters only one *full* paraproduct – not four. In particular, some demanding aspects related to mixed full paraproducts arising from partial adjoints of Calderón–Zygmund operators are not present here.

Recently, the author together with M. Mourgoglou [7] proved a boundedness criterion for one-parameter square functions with general measures. The key to the short proof was based on a new averaging identity over good Whitney regions. The identity is a further development of Hytönen's improvement [3] of the Nazarov–Treil–Volberg method of random dyadic systems [8]. In this paper we push this efficient proof strategy to the much more demanding case of two parameters. Our boundedness proof is surprisingly short compared to most of the related bi-parameter theory.

Probabilistic methods in the bi-parameter Calderón–Zygmund setting were first used by the author in [6]. They saw another application in a joint work with Hytönen [4]. Most recently these methods were used by Y. Ou [9] to prove a Tb extension of [6]. However, the square function case treated in this paper is based on a simple new averaging identity over good double Whitney regions. Even in the probabilistic realm the square function case is cleaner than the corresponding Calderón–Zygmund case.

The first $T1$ type theorem for product spaces was proved by Journé [5]. Journé formulated his theorem in the language of vector-valued Calderón–Zygmund theory. S. Pott and P. Villarroya [10] recently offered a new view – an alternative framework avoiding the vector-valued assumptions. This ideology of mixing the various conditions (kernel estimates, BMO and weak boundedness property) was also used in [6] and [4]. The current paper is a continuation of this theme but in the square function setting. For the corresponding one-parameter square function theory see e.g. the papers by Christ–Journé [1], Hofmann [2] and Semmes [11].