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Elliptic differential operators on Lipschitz domains and abstract boundary value problems

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ABSTRACT

This paper consists of two parts. In the first part, which is of more abstract nature, the notion of quasi-boundary triples and associated Weyl functions is developed further in such a way that it can be applied to elliptic boundary value problems on non-smooth domains. A key feature is the extension of the boundary maps by continuity to the duals of certain range spaces, which directly leads to a description of all self-adjoint extensions of the underlying symmetric operator with the help of abstract boundary values. In the second part of the paper a complete description is obtained of all self-adjoint realizations of the Laplacian on bounded Lipschitz domains, as well as Kreĭn type resolvent formulas and a spectral characterization in terms of energy dependent Dirichlet-to-Neumann maps. These results can be viewed as the natural generalization of recent results by Gesztesy and Mitrea for quasi-convex domains. In this connection we also characterize the maximal range spaces of the Dirichlet and Neumann trace operators on a bounded Lipschitz domain in terms of the Dirichlet-to-Neumann map. The general results from the first part of the paper are also applied to higher order elliptic operators

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on smooth domains, and particular attention is paid to the second order case which is illustrated with various examples.

1. Introduction

Spectral theory of elliptic partial differential operators has received a lot of attention in the recent past, in particular, modern techniques from abstract operator theory were applied to extension and spectral problems for symmetric and self-adjoint elliptic differential operators on bounded and unbounded domains. We refer the reader to the recent contributions [3,11–13,17,18,43–45,53] on smooth domains, [1,4,5,33–35,40,42,61,62,64] on non-smooth domains, and we point out the paper [36] by Gesztesy and Mitrea which has inspired parts of the present work. Many of these contributions are based on the classical works Grubb [39] and Višik [72] on the parameterization of the closed realizations of a given elliptic differential expression on a smooth domain, and other classical papers on realizations with local and non-local boundary conditions, see, e.g. [2,8,9,16,32,68] and the monograph [52] by Lions and Magenes.

In [36] Gesztesy and Mitrea obtain a complete description of the self-adjoint realizations of the Laplacian on a class of bounded non-smooth, so-called *quasi-convex* domains. The key feature of quasi-convex domains is that the functions in the domains of the self-adjoint Dirichlet realization Δ_D and the self-adjoint Neumann realization Δ_N possess H^2-regularity, a very convenient property which is well-known to be false for the case of Lipschitz domains; cf. [49]. Denote by τ_D and τ_N the Dirichlet and Neumann trace operator, respectively. Building on earlier work of Maz'ya, Mitrea and Shaposhnikova [55], see also [21,31,37], the range spaces $\mathscr{G}_0 := \tau_D(\operatorname{dom} \Delta_N)$ and $\mathscr{G}_1 := \tau_N(\operatorname{dom} \Delta_D)$ were characterized for quasi-convex domains in [36], and the self-adjoint realizations of the Laplacian were parameterized via tuples $\{\mathscr{X}, L\}$, where $\mathscr{X}$ is a closed subspace of the anti-dual $\mathscr{G}_0'$ or $\mathscr{G}_1'$ and L is a self-adjoint operator from $\mathscr{X}$ to $\mathscr{X}'$. This parameterization technique has its roots in [15,51] and was used in [39,72], see also [41, Chapter 13]. In [17] the connection to the notion of (ordinary) boundary triples from extension theory of symmetric operators was made explicit.

The theory of ordinary boundary triples and Weyl functions originates in the works of Kočubeĭ [50], Bruk [19], Gorbachuk and Gorbachuk [38], and Derkach and Malamud [27,28]. A boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for a symmetric operator A in a Hilbert space $\mathcal{H}$ consists of an auxiliary Hilbert space $\mathcal{G}$ and two boundary mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} A^* \to \mathcal{G}$ which satisfy an abstract Green's identity and a maximality condition. With the help of a boundary triple the closed extensions of the underlying symmetric operator A can be parameterized in an efficient way with closed operators and subspaces Θ in the boundary space $\mathcal{G}$. The concept of ordinary boundary triples was applied successfully to various problems in extension and spectral theory, in particular, in the context of ordinary differ-