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Generalized Poincaré–Hopf theorem and application to nonlinear elliptic problem

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A B S T R A C T

In this paper we get an extended version of Poincaré–Hopf theorem. Without the assumption that critical point set between two level sets of energy functional is finite, this result actually generalizes Morse inequality. And the isomorphism between $C_q(J,\infty)$ and $C_q(J_{(a,b)},0)$ is yielded as $(a,b)\notin\Sigma$, where $C_q(J,\infty)$ denotes the critical groups of energetic functional J at infinity and $C_q(J_{(a,b)},0)$ stands for the critical groups of functional $J_{(a,b)}$ at zero, and Σ is the set of points $(a,b)\in\mathbb{R}^2$ for which the problem

$$\begin{cases} -\Delta u+\alpha u=au^-+bu^+, & x\in\Omega,\\ \frac{\partial u}{\partial\nu}=0, & x\in\partial\Omega, \end{cases}$$

has a nontrivial solution, $u^+=\max\{u,0\}$, $u^-=\min\{u,0\}$. (Concerning the definitions of J and $J_{(a,b)}$, see Section 4.) As to application aspects, we are mainly concerned with nonlinear elliptic problem with Neumann boundary condition provided that the origin is a non-isolated critical point and obtain the existence of multiple solutions.

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1. Introduction

The celebrated Poincaré–Hopf theorem shows us the relationship between the indices of zeros of a smooth vector field on a manifold M and the Euler Characteristic of M. The theorem was proven for two dimensions by Henri Poincaré in 1885 and later generalized to higher dimensions by Heinz Hopf [7] in 1926. Infinite dimensional case was studied by E. Rothe [15] and he got the following (1.1) under stronger assumptions. Subsequently, K.C. Chang verified Proposition 1.1 below (see [4] for details):

Proposition 1.1. *Let E be a real Hilbert space and $f \in C^2(E, \mathbb{R})$ be a function that satisfies the (PS) condition. Assume that*

$$df(u) = u - K(u)$$

and u_0 is an isolated critical point of f on E. Then we have

$$\begin{aligned} \deg\bigl(df, B(u_0, \varepsilon), 0\bigr) &= \deg\bigl(I - K, B(u_0, \varepsilon), 0\bigr) \\ &= \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(f, u_0) \end{aligned} \tag{1.1}$$

for $\varepsilon > 0$ sufficiently small, where $\deg(df, B(u_0, \varepsilon), 0)$ stands for the Leray–Schauder degree of df at u_0, $B(u_0, \varepsilon) := \{u \in E : \|u - u_0\| < \varepsilon\}$, I denotes identity map and K is a compact mapping, and

$$C_q(f, u_0) := H_q\bigl(f^c \cap B(u_0, \varepsilon), \bigl(f^c \backslash \{u_0\}\bigr) \cap B(u_0, \varepsilon), G\bigr)$$

stands for the q-th critical group, with coefficient group G of f at u_0, $c = f(u_0)$, $f^c := \{u \in E : f(u) \le c\}$, and $H_*(X, Y, G)$ represents the singular relative homology groups with the abelian coefficient group G.

K.C. Chang also generalized Proposition 1.1 as follows (see [4]):

Proposition 1.2. *Let E be a real Hilbert space. Suppose $f \in C^2(E, \mathbb{R})$ is a function that satisfies the (PS) condition. Assume that W is a bounded domain in E on which f is bounded. Assume that*

(a) $W_- := \{u \in \partial W : \eta(t, u) \notin W, \ \forall t > 0\} = W \cap f^{-1}(a)$ for some a, where $\eta(t, u)$ is negative gradient flow of f emanating from u;
(b) $-df$ points inward at $\partial W \backslash W_-$, then we have

$$\deg(df, W, 0) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} H_q(W, W_-). \tag{1.2}$$