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A new characterization of the Clifford torus via scalar curvature pinching ☆

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A R T I C L E I N F O

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A B S T R A C T

Let M^n be a compact hypersurface with constant mean curvature H in $\mathbb{S}^{n+1}$. Denote by S the squared norm of the second fundamental form of M. We prove that there exists an explicit positive constant $\gamma(n)$ depending only on n such that if $|H| \le \gamma(n)$ and $\beta(n,H) \le S \le \beta(n,H) + \frac{n}{23}$, then $S \equiv \beta(n,H)$ and M is one of the following cases: (i) $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \le k \le n-1$; (ii) $\mathbb{S}^1(\frac{1}{\sqrt{1+\mu^2}}) \times \mathbb{S}^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$. Here $\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$ and $\mu = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2}$. For $n \in \{6, 12, 24\}$, we give examples to show that the assumption $|H| \le \gamma(n)$ cannot be removed.

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1. Introduction

Let M^n be an n-dimensional compact hypersurface with constant mean curvature in an $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$. Denote by R, H and S the scalar curvature, the mean curvature and the squared norm of the second fundamental form of M, respectively. It follows from the Gauss equation that $R = n(n-1) + n^2H^2 - S$. A famous rigidity theorem due to Simons, Lawson, and Chern, do Carmo and Kobayashi [14,19,28] says that if M is a closed minimal hypersurface in $\mathbb{S}^{n+1}$ satisfying $S \le n$, then $S \equiv 0$ and M is the great sphere $\mathbb{S}^n$, or $S \equiv n$ and M is the Clifford torus $\mathbb{S}^k(\sqrt{\frac{k}{n}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \le k \le n-1$. Afterwards, Li–Li [21] improved Simons' pinching constant for n-dimensional closed minimal submanifolds in $\mathbb{S}^{n+p}$ to $\max\{\frac{n}{2-1/p}, \frac{2}{3}n\}$. Further developments on this rigidity theorem have been made by many other authors [13,16,22,23,32,33,39], etc. In 1970's, Chern proposed the following conjecture.

Chern Conjecture. *(See [14,40].) Let M be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.*

(A) *If S is constant, then the possible values of S form a discrete set. In particular, if $n \le S \le 2n$, then $S = n$, or $S = 2n$.*
(B) *If $n \le S \le 2n$, then $S \equiv n$, or $S \equiv 2n$.*

In 1983, Peng and Terng [26,27] made a breakthrough on the Chern conjecture, and proved the following

Theorem A. *Let M be a compact minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.*

(i) *If S is constant, and if $n \le S \le n + \frac{1}{12n}$, then $S = n$.*
(ii) *If $n \le 5$, and if $n \le S \le n + \tau_1(n)$, where $\tau_1(n)$ is a positive constant depending only on n, then $S \equiv n$.*

During the past three decades, there have been some important progress on these aspects [7,11,12,17,29,30,37,38,41], etc. In 1993, Chang [7] proved Chern Conjecture (A) in dimension three. Yang–Cheng [38] improved the pinching constant $\frac{1}{12n}$ in Theorem A(i) to $\frac{n}{3}$. Later, Suh–Yang [29] improved this pinching constant to $\frac{3}{7}n$.

In 2007, Wei and Xu [30] proved that if M is a compact minimal hypersurface in $\mathbb{S}^{n+1}$, $n = 6, 7$, and if $n \le S \le n + \tau_2(n)$, where $\tau_2(n)$ is a positive constant depending only on n, then $S \equiv n$. Later, Zhang [41] extended the second pinching theorem due to Peng–Terng [27] and Wei–Xu [30] to the case of $n = 8$. Recently, Ding and Xin [17] obtained the striking result, as stated