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# On the impossibility of $W_p^2$ estimates for elliptic equations with piecewise constant coefficients

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In this paper, we present counterexamples showing that for any $p \in (1, \infty)$, $p \neq 2$, there is a non-divergence form uniformly elliptic operator with piecewise constant coefficients in $\mathbb{R}^2$ (constant on each quadrant in $\mathbb{R}^2$) for which there is no $W_p^2$ estimate. The corresponding examples in the divergence case are also discussed. One implication of these examples is that the ranges of $p$ are sharp in the recent results obtained in [4,5] for non-divergence type elliptic and parabolic equations in a half space with the Dirichlet or Neumann boundary condition when the coefficients do not have any regularity in a tangential direction.



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## 1. Introduction and main results

We consider elliptic operators in non-divergence form

$$\mathcal{L}u = a^{ij} D_{ij} u,$$

where

$$\delta|\xi|^2 \le a^{ij}\xi_i\xi_j, \quad \left|a^{ij}\right| \le \delta^{-1}, \ \delta \in (0,1]. \tag{1.1}$$

The $L_p$-theory of second-order elliptic and parabolic equations with discontinuous coefficients has been studied extensively in the last fifty years. In the special case when the dimension $d = 2$, it is well known that the $W_2^2$ estimate holds for uniformly elliptic operators with general bounded and measurable coefficients. See, for instance, [2,20]. On the other hand, a celebrated counterexample in [19] and [15] indicates that when $d \ge 3$ in general there is no $W_2^2$ estimate for elliptic operators with bounded measurable coefficients even if they are discontinuous only at a single point. Another example due to Ural'tseva [21] (see also [12]) shows the impossibility of the $W_p^2$ estimate when $d \ge 2$ and $p \ne 2$. We note that in Ural'tseva's example, the coefficients are continuous except at a single point ($d = 2$) or a line ($d = 3$). In [16], Nadirashvili showed that the weak uniqueness for martingale problems may fail if coefficients are merely measurable and $d \ge 3$. These examples imply that in general there does not exist a solvability theory for uniformly elliptic operators with bounded and measurable coefficients. Thus many efforts have been made to treat particular types of discontinuous coefficients.

In [3] Campanato extended the aforementioned result in [2,20] to the case when $d = 2$ and $p$ is in a neighborhood of 2, the size of which depends on the ellipticity constant $\delta$. A corresponding result for parabolic equations can be found in [11]. By using explicit representation formulae, Lorenzi [13,14] studied the $W_2^2$ and $W_p^2$, $1 < p < \infty$, estimates for elliptic equations in $\mathbb{R}^d$ with coefficients which are constant on each half space. See [18,8] for similar results for parabolic equations, and [9] for elliptic equations in $\mathbb{R}^d$ with leading coefficients discontinuous at finitely many parallel hyperplanes. We also refer the reader to [10,6,12,4,5] and the references therein for some recent developments for equations with coefficients only measurable in some directions. In particular, it is proved in [4] that the $W_p^2$ estimate holds for elliptic equations in a half space with the zero Dirichlet (or Neumann) boundary condition when coefficients are only measurable in a tangential direction to the boundary and $p \in (1,2]$ (or $p \in [2,\infty)$, respectively).

In this paper we focus our attention to elliptic equations with *piecewise constant* coefficients in $\mathbb{R}^2$. In fact, the results in [14,10] imply the $W_p^2$, $1 < p < \infty$, estimate for such equations if coefficients are constants on the upper half plane and another constants on the lower half plane. On the other hand, as a special case of the results in [4], we have the $W_p^2$, $1 < p \le 2$ (or $2 \le p < \infty$), estimate for equations defined in the upper half plane with the Dirichlet (or Neumann, respectively) boundary condition if coefficients