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Maximality and numéraires in convex sets of nonnegative random variables ☆

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A R T I C L E I N F O

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A B S T R A C T

We introduce the concepts of max-closedness and numéraires of convex subsets of $\mathbb{L}^0_+$, the nonnegative orthant of the topological vector space $\mathbb{L}^0$ of all random variables built over a probability space, equipped with a topology consistent with convergence in probability. Max-closedness asks that maximal elements of the closure of a set already lie on the set. We discuss how numéraires arise naturally as strictly positive optimisers of certain concave monotone maximisation problems. It is further shown that the set of numéraires of a convex, max-closed and bounded set of $\mathbb{L}^0_+$ that contains at least one strictly positive element is dense in the set of its maximal elements.

0. Introduction

0.1. Discussion

Let $\mathbb{L}^0$ denote the set of all (equivalence classes of real-valued) random variables built over a probability space, equipped with a metric topology under which convergence of sequences coincides with convergence in probability. Denote by $\mathbb{L}^0_+$ the nonnegative orthant

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of $\mathbb{L}^0$. In many problems of interest—notably, in the field of mathematical finance—one seeks maximisers of a concave and strictly monotone (increasing) functional $\mathbb{U}$ over convex set $\mathcal{C} \subseteq \mathbb{L}^0_+$. In order to ensure that such optimisers exist, some closedness property of $\mathcal{C}$ should be present. The strict monotonicity of $\mathbb{U}$ *a priori* implies that, if optimisers exist, they must be maximal elements of $\mathcal{C}$ with respect to the natural lattice structure of $\mathbb{L}^0$; therefore, a natural condition to enforce is that maximal points of the closure of $\mathcal{C}$ already lie in $\mathcal{C}$. We then refer to the set $\mathcal{C}$ as being *max-closed*, and the collection $\mathcal{C}^{\mathsf{max}}$ of all its maximal elements is regarded as the "outer boundary" of $\mathcal{C}$.

Concave maximisation problems as the one described above are particularly amenable to first-order analysis. Morally speaking, a maximiser of a concave functional $\mathbb{U}$ over $\mathcal{C}$ should also be a maximiser of a *nice* nonzero linear functional over $\mathcal{C}$. When *nice* means *continuous*, such an element is called a *support point* of $\mathcal{C}$ in traditional functional-analytic framework, and existence of a supporting nonzero continuous linear functional is typically provided by an application of the geometric form of the Hahn–Banach theorem. Unfortunately, $\mathbb{L}^0$ is rather unsuitable[1] for application of standard convex-analytic techniques. More precisely, when the probability space is non-atomic:

- $\mathbb{L}^0$ fails to be locally convex, which implies that a rich body of results (including the Hahn–Banach theorem) cannot be used;
- the topological dual of $\mathbb{L}^0$ contains only the zero functional [5, Theorem 2.2, p. 18]; in particular, as the Namioka–Klee theorem [11] suggests, there is no real-valued nonzero positive linear functional on $\mathbb{L}^0$.

In particular, convex sets in $\mathbb{L}^0$ *a fortiori* lack support points according to the usual definition. The previous issue notwithstanding, this work aims at exploring special elements of convex subsets of $\mathbb{L}^0_+$ which can be regarded as support points. More precisely, we discuss the notion of a *numéraire* g of a set $\mathcal{C} \subseteq \mathbb{L}^0_+$, asking that g is strictly positive (in the sense that $\{g = 0\}$ is a null set) and there exists a probability measure $\mathbb{Q}$, equivalent to the underlying probability measure, such that $\mathbb{E}_{\mathbb{Q}}[f/g] \leq 1$ holds for all $f \in \mathcal{C}$, where "$\mathbb{E}_{\mathbb{Q}}$" denotes expectation under $\mathbb{Q}$. As is argued in the article (see Remark 2.3), numéraires are closely related to support points in the classical sense, where the supporting "dual element" corresponds to a σ-additive, σ-finite, positive measure, equivalent to the underlying probability measure. Furthermore, by means of the rather wide-encompassing example in Section 2.2 it is rigorously illustrated that optimisers for a large class of concave monotone maximisation problems over convex sets are indeed numéraires according to the previous definition.

[1] Note, however, that whenever $\mathcal{C} \subseteq \mathbb{L}^0_+$ is a convex and bounded (in measure) set, there exists a probability $\mathbb{Q}$, equivalent to the underlying one, such that $\mathcal{C}$ is bounded in $\mathbb{L}^1(\mathbb{Q})$—see discussion after Theorem 3.1; although this sometimes facilitates the analysis on $\mathcal{C}$, in general the $\mathbb{L}^0$-topology does not coincide with the $\mathbb{L}^1(\mathbb{Q})$-topology on $\mathcal{C}$.