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Homogenization of a generalized Stefan problem in the context of ergodic algebras

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ABSTRACT

We address the deterministic homogenization, in the general context of ergodic algebras, of a doubly nonlinear problem which generalizes the well known Stefan model, and includes the classical porous medium equation. It may be represented by the differential inclusion, for a real-valued function $u(x,t)$,

$$\frac{\partial}{\partial t}\partial_u \Psi(x/\varepsilon, x, u) - \nabla_x \cdot \nabla_\eta \psi(x/\varepsilon, x, t, u, \nabla u) \ni f(x/\varepsilon, x, t, u),$$

on a bounded domain $\Omega \subseteq \mathbb{R}^n$, $t \in (0,T)$, together with initial–boundary conditions, where $\Psi(z,x,\cdot)$ is strictly convex and $\psi(z,x,t,u,\cdot)$ is a C^1 convex function, both with quadratic growth, satisfying some additional technical hypotheses. As functions of the oscillatory variable, $\Psi(\cdot,x,u), \psi(\cdot,x,t,u,\eta)$ and $f(\cdot,x,t,u)$ belong to the generalized Besicovitch space $\mathcal{B}^2$ associated with an arbitrary ergodic algebra $\mathcal{A}$. The periodic case was addressed by Visintin (2007), based on the two-scale convergence technique. Visintin's analysis for the periodic case relies heavily on the possibility of reducing two-scale convergence to the usual L^2 convergence in the Cartesian product $\Pi \times \mathbb{R}^n$, where Π is the periodic cell. This reduction is no longer possible in the case of a general ergodic algebra. To overcome this difficulty, we make essential use of the concept

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of two-scale Young measures for algebras with mean value, associated with bounded sequences in L^2.

1. Introduction

In this paper we are concerned with the homogenization of the following initial–boundary value problem, on $\Omega \times (0,T)$, with $\Omega \subseteq \mathbb{R}^n$ a bounded open set,

$$\begin{aligned}
&\partial_t w_\varepsilon - \nabla \cdot \alpha(\frac{x}{\varepsilon}, x, t, u_\varepsilon, \nabla u_\varepsilon) = f(\frac{x}{\varepsilon}, x, u_\varepsilon),\\
&w_\varepsilon(x,t) \in \partial\Psi(\frac{x}{\varepsilon}, x, u_\varepsilon), \qquad \text{a.e. in } \Omega \times (0,T),\\
&w_\varepsilon(x,0) = w_0(\frac{x}{\varepsilon}, x), \qquad x \in \Omega,\\
&u_\varepsilon = 0, \qquad \text{in } \partial\Omega \times (0,T).
\end{aligned} \tag{1.1}$$

Here $\Psi(z,x,\cdot)$ is a strictly convex function for a.e. $(z,x) \in \mathbb{R}^n \times \Omega$, $\partial\Psi(z,x,v)$ denotes the subdifferential of Ψ with respect to v, and for all $v \in \mathbb{R}$, $\alpha(z,x,t,v,\cdot)$ is a monotone operator which is also assumed to satisfy a coercivity condition such as $\alpha(z,x,t,v,\eta)\cdot\eta \geq c_0|\eta|^2 + h(x,t)$, with $c_0 > 0$ and $h \in L^1(\Omega \times (0,T))$, as usual. Also, $\alpha(z,x,t,v,\eta) = \nabla_\eta \psi(z,x,t,v,\eta)$, where $\psi(z,x,t,v,\cdot)$ is C^1 and convex in $\mathbb{R}^n$, for all $v \in \mathbb{R}$ and a.e. $(z,x,t) \in \mathbb{R}^n \times \Omega \times (0,T)$.

We address this homogenization problem in the context of a general ergodic algebra $\mathcal{A}$ on $\mathbb{R}^n$. The latter is a concept introduced by Zhikov and Krivenko in [42], which largely extends the class of almost periodic functions, introduced by Bohr (see, e.g., [7]). More generally, it abstracts the properties satisfied by realizations of continuous functions in general compact topological spaces, endowed with a probability measure μ, under the action of a continuous dynamical system for which μ is invariant (*cf.* [3], and Theorem A.10 in Appendix A). It includes the almost periodic functions, already mentioned, as well as the more general weak almost periodic functions, introduced by Eberlein, which strictly include the Fourier–Stieltjes transforms (see, e.g., [17], and the discussion in Appendix A, below).

Let $\mathcal{B}^2$ denote the generalized Besicovitch space associated with the ergodic algebra $\mathcal{A}$, with topology provided by the semi-norm given as the mean value of the square of the absolute value (see Appendix A). We assume that, for all $v \in \mathbb{R}$, $\Psi(\cdot,\cdot,v) \in \mathcal{B}^2(\mathbb{R}^n; L(\Omega))$. Further, $\alpha(\cdot,\cdot,\cdot,v,\eta) \in \mathcal{B}^2(\mathbb{R}^n; L^2(\Omega \times (0,T))$ and $f(\cdot,\cdot,v) \in \mathcal{B}^2(\mathbb{R}^n; L^2(\Omega))$, for all $v \in \mathbb{R}$ and all $\eta \in \mathbb{R}^n$. Additional technical conditions are assumed over Ψ, ψ, α, f, see **(Ψ1)**–**(Ψ4)**, **(ψ1)**–**(ψ4)**, **(α1)**–**(α5)**, in Section 3, and **(f1)**–**(f3)** in Section 4, below.

The corresponding case where $\mathcal{A}$ is the algebra of the continuous periodic functions on $\mathbb{R}^n$ was addressed in the pioneering paper by Visintin, [40], and the main point of the