Journal of Functional Analysis 268 (2015) 3404–3477

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Intrinsic pseudo-differential calculi on any compact Lie group

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A R T I C L E I N F O

Article history:
Received 25 September 2014
Accepted 22 March 2015
Available online 29 March 2015
Communicated by L. Saloff-Coste

MSC:
43A75
22E30
58J40
35S05

Keywords:
Harmonic analysis on Lie groups
Analysis on compact Lie groups
Pseudo-differential calculus on manifolds

A B S T R A C T

In this paper, we define in an intrinsic way operators on a compact Lie group by means of symbols using the representations of the group. The main purpose is to show that these operators form a symbolic pseudo-differential calculus which coincides or generalises the (local) Hörmander pseudo-differential calculus on the group viewed as a compact manifold.

1. Introduction

Over the past five decades, pseudo-differential operators have become a powerful and versatile tool in the analysis of Partial Differential Equations (PDE's) in various contexts. Although they may be used for global analysis (essentially in the Euclidean setting), they

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http://dx.doi.org/10.1016/j.jfa.2015.03.015

can be localised and this allows one to define them on closed manifolds. However, on a closed manifold, one can no longer attach a global symbol to a single operator in the calculus (although one could recover a – partial – global definition of operators on manifolds for instance using linear connections, see [13] and the references therein). The subject of the present paper is to define globally and intrinsically symbolic calculi on a special class of manifolds, more precisely on any compact Lie group G. Naturally the first aim of this article is to show that the fundamental properties of the calculi hold true, thereby justifying the vocabulary. The second aim of this article is to prove that our calculi coincide with the Hörmander calculi localised on G viewed as a compact manifold – when the Hörmander calculi can be defined. We will also show that it coincides with the calculi proposed by Michael Ruzhansky and Ville Turunen in [10]. Although this is not the purpose of this paper, let us mention that several applications to PDE's of the calculi have been obtained by Michael Ruzhansky, Ville Turunen and Jens Wirth, e.g. construction of parametrices, study of global hypoellipticity, see [12,10] and the references therein.

It is quite natural to define pseudo-differential operators globally on the torus by using Fourier series and considering symbols as functions of a variable in the torus and another variable in the integer lattice, see for instance [11] and the references therein. Michael Taylor argued in his monograph [17, Section I.2] that an analogue quantisation is formally true on any Lie group of type 1, considering again symbols as functions of a variable of the group G and another variable of its dual $\widehat{G}$ (which is the set of equivalence classes of the unitary irreducible representations of G). Just afterwards, Zelditch in [19] defined a (compactly-supported) symbolic pseudo-differential calculus on a hyperbolic manifold with a related quantisation. Pseudo-differential calculi have also been defined on the Heisenberg group by Taylor in [17], see also [2] and [6], and in other directions by Dynin, Folland, Beals, Greiner, Howe (see [7] and the references therein). See also [4] for a global pseudo-differential calculus on homogeneous Lie groups (although it may not qualify as symbolic, being defined in terms of properties of the kernels of the operators).

It would be nearly impossible to review in this introduction the vast literature on classes of operators defined on Lie groups (especially if one has to include all the studies of spectral multipliers of sub-Laplacians). Instead, in this article, we focus on pseudo-differential operators, in the sense that the operators are not necessarily of convolution type. In this sense, studies of pseudo-differential calculi on Lie groups form a much shorter list and the ones known to the author were mentioned directly or indirectly earlier in this introduction.

Following the ideas in the introduction of [2], let us formalise what is meant here by a calculus:

Definition 1.1. For each $m \in \mathbb{R}$, let Ψ^m be a given Fréchet space of continuous operators $\mathcal{D}(G) \to \mathcal{D}(G)$. We say that the space $\Psi^\infty := \cup_m \Psi^m$ form a *pseudo-differential calculus* when it is an algebra of operators satisfying: