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Exponential bounds for the support convergence in the Single Ring Theorem

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A B S T R A C T

We consider an $n \times n$ matrix of the form $\mathbf{A} = \mathbf{UTV}$, with $\mathbf{U}$, $\mathbf{V}$ some independent Haar-distributed unitary matrices and $\mathbf{T}$ a deterministic matrix. We prove that for $k \sim n^{1/6}$ and $b^2 := \frac{1}{n}\operatorname{Tr}(|\mathbf{T}|^2)$, as n tends to infinity, we have

$$\mathbb{E}\operatorname{Tr}(\mathbf{A}^k(\mathbf{A}^k)^*) \lesssim b^{2k} \qquad \text{and} \qquad \mathbb{E}[|\operatorname{Tr}(\mathbf{A}^k)|^2] \lesssim b^{2k}.$$

This gives a simple proof (with slightly weakened hypothesis) of the convergence of the support in the Single Ring Theorem, improves the available error bound for this convergence from $n^{-\alpha}$ to $e^{-cn^{1/6}}$ and proves that the rate of this convergence is at most $n^{-1/6}\log n$.

1. Introduction

The Single Ring Theorem, by Guionnet, Krishnapur and Zeitouni [8], describes the empirical distribution of the eigenvalues of a large generic matrix with prescribed singular values, *i.e.* an $n \times n$ matrix of the form $\mathbf{A} = \mathbf{UTV}$, with $\mathbf{U}$, $\mathbf{V}$ some independent Haar-distributed unitary matrices and $\mathbf{T}$ a deterministic matrix whose singular values are

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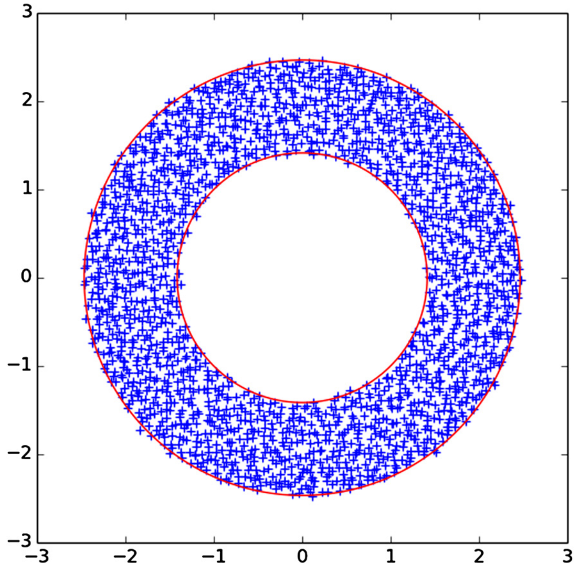

Fig. 1. Spectrum of $\mathbf{A}$ when the s_i's are uniformly distributed on $[0.5, 4]$, so that $a \approx 1.41$ and $b \approx 2.47$ (here, $n = 2.10^3$).

the ones prescribed. More precisely, under some technical hypotheses,[1] as the dimension n tends to infinity, if the empirical distribution of the singular values of $\mathbf{A}$ converges to a compactly supported limit measure Θ on the real line, then the empirical eigenvalues distribution of $\mathbf{A}$ converges to a limit measure μ on the complex plane which depends only on Θ. The limit measure μ (see Fig. 1) is rotationally invariant in $\mathbb{C}$ and its support is the annulus $\{z \in \mathbb{C}\,;\, a \le |z| \le b\}$, with $a, b \ge 0$ such that

$$a^{-2} = \int x^{-2}\mathrm{d}\Theta(x) \qquad \text{and} \qquad b^2 = \int x^2\mathrm{d}\Theta(x). \tag{1}$$

In [9], Guionnet and Zeitouni also proved the convergence in probability of the support of the empirical eigenvalues distribution of $\mathbf{A}$ to the support of μ. The reason why the radii a and b of the borders of the support of μ are given by (1) is related to the earlier work [10] by Haagerup and Larsen about R-diagonal elements in free probability theory but has no simple explanation: the matrix $\mathbf{A}$ is far from being normal, hence its spectral radius should be smaller than its operator norm, *i.e.* than the L^∞-norm[2] of Θ, but, up to our knowledge, there is no evidence why this modulus has to be close to the L^2-norm of Θ, as follows from (1).

Another way to see the problem is the following one. In [19], Rudelson and Vershynin have proved that there is a universal constant c such that the smallest singular value

[1] These hypotheses have been weakened by Rudelson and Vershynin in [19] and by Basak and Dembo in [1].

[2] To be precise, we should say the "L^∞-norm of *a* Θ*-distributed r.v.*" rather than "L^∞-norm of Θ". The same is true for the L^2-norm hereafter.