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Scattering in twisted waveguides

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Abstract

We consider a twisted quantum waveguide, i.e. a domain of the form $\Omega_\theta := r_\theta \omega \times \mathbb{R}$ where $\omega \subset \mathbb{R}^2$ is a bounded domain, and $r_\theta = r_\theta(x_3)$ is a rotation by the angle $\theta(x_3)$ depending on the longitudinal variable x_3. We investigate the nature of the essential spectrum of the Dirichlet Laplacian $\mathcal{H}_\theta$, self-adjoint in $\mathrm{L}^2(\Omega_\theta)$, and consider related scattering problems. First, we show that if the derivative of the difference $\theta_1 - \theta_2$ decays fast enough as $|x_3| \to \infty$, then the wave operators for the operator pair $(\mathcal{H}_{\theta_1}, \mathcal{H}_{\theta_2})$ exist and are complete. Further, we concentrate on appropriate perturbations of constant twisting, i.e. $\theta' = \beta - \varepsilon$ with constant $\beta \in \mathbb{R}$, and ε which decays fast enough at infinity together with its first derivative. In that case the unperturbed operator corresponding to ε is an analytically fibered Hamiltonian with purely absolutely continuous spectrum. Obtaining Mourre estimates with a suitable conjugate operator, we prove, in particular, that the singular continuous spectrum of $\mathcal{H}_\theta$ is empty.

Keywords: Twisted waveguides; Wave operators; Mourre estimates

1. Introduction

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial\omega \in C^2$. Denote by $\Omega := \omega \times \mathbb{R}$ the straight tube in $\mathbb{R}^3$. For a given $\theta \in C^1(\mathbb{R}, \mathbb{R})$ we define the twisted tube Ω_θ by

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$$\Omega_\theta = \{ r_\theta(x_3)x \in \mathbb{R}^3 \mid x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ x_\omega := (x_1, x_2) \in \omega \},$$

where

$$r_\theta(x_3) = \begin{pmatrix} \cos\theta(x_3) & \sin\theta(x_3) & 0 \\ -\sin\theta(x_3) & \cos\theta(x_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We define the Dirichlet Laplacian $\mathcal{H}_\theta$ as the unique self-adjoint operator generated in $\mathrm{L}^2(\Omega_\theta)$ by the closed quadratic form

$$\mathcal{Q}_\theta[u] := \int_{\Omega_\theta} |\nabla u|^2 \, d\mathbf{x}, \quad u \in \mathrm{D}(\mathcal{Q}_\theta) := \mathrm{H}_0^1(\Omega_\theta). \tag{1.1}$$

In fact, we do not work directly with $\mathcal{H}_\theta$, but rather with a unitarily equivalent operator $H_{\theta'}$ acting in the straight tube Ω, see (2.4). The related unitary transformation is generated by a change of variables which maps the twisted tube Ω_θ onto the straight tube Ω, see Eq. (2.3).

The goal of the present article is to study the nature of the essential spectrum of the operator $\mathcal{H}_\theta$ under appropriate assumptions about the twisting angle θ. Although the spectral properties of a twisted waveguide have been intensively studied in recent years, attention has been paid mostly to the discrete spectrum of $\mathcal{H}_\theta$, [5,10,14], or to the Hardy inequality for $\mathcal{H}_\theta$, [9].

In this article we discuss the influence of twisting on the nature of the essential spectrum of $\mathcal{H}_\theta$. First, we show that if the difference $\theta_1' - \theta_2'$ decays fast enough as $|x_3| \to \infty$, then the wave operators for the operator pair $(H_{\theta_1'}, H_{\theta_2'})$ exist and are complete, and in particular, the absolutely continuous spectra of $H_{\theta_1'}$ and $H_{\theta_2'}$ coincide. Further, we observe that if $\theta' = \beta$ is constant, then the operator H_β is analytically fibered, cf. (2.9), and therefore its singular continuous spectrum is empty, [11,13]. Assuming that $\theta'(x_3) = \beta - \varepsilon(x_3)$ with $\varepsilon \in C^1(\mathbb{R}, \mathbb{R})$, we then show that if ε decays fast enough at infinity, then $H_{\theta'}$ has no singular continuous spectrum, see Theorem 2.7. The proof of Theorem 2.7 is based on the Mourre commutator method, [19,20,1]. We construct a suitable conjugate operator A and show that the commutator $[H_{\theta'}, iA]$ satisfies a Mourre estimate on sufficiently small intervals outside a discrete subset of $\mathbb{R}$, Theorem 8.2. The construction of the conjugate operator is based on a careful analysis of the band functions $E_n(k)$ of the unperturbed operator H_β, $k \in \mathbb{R}$ being the Fourier variable dual to x_3. A similar strategy was used in [12,2,6,17], where the generator of dilations in the longitudinal direction of the waveguide was used as a conjugate operator. However, in the situations studied in these works the associated band functions have a non-zero derivative everywhere except for the origin. In our model, contrary to [12,2,6,17], the band functions E_n may have many stationary points. In addition, we have to take into account possible crossing points between different band functions. The generator of dilations therefore cannot be used as a conjugate operator in our case, and a different approach is needed. Our conjugate operator acts in the fibered space as

$$\frac{i}{2}\bigl(\gamma(k)\partial_k + \partial_k\gamma(k)\bigr) \tag{1.2}$$

where $\gamma \in C_0^\infty(\mathbb{R}; \mathbb{R})$ is a suitably chosen function, whose particular form depends on the interval on which the Mourre estimate is established, see Theorem 7.2.