Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 266 (2014) 36–54

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Dual Lukacs regressions for non-commutative variables

Kamil Szpojankowski *, Jacek Wesołowski

Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska, ul. Koszykowa 75, 00-662 Warszawa, Poland

Received 21 October 2011; accepted 19 September 2013

Available online 4 October 2013

Communicated by D. Voiculescu

Abstract

Dual Lukacs type characterizations of random variables in free probability are studied here. First, we develop a freeness property satisfied by Lukacs type transformations of free-Poisson and free-binomial non-commutative variables which are free. Second, we give a characterization of non-commutative free-Poisson and free-binomial variables by properties of first two conditional moments, which mimic Lukacs type assumptions known from classical probability. More precisely, our result is a non-commutative version of the following result known in classical probability: if U, V are independent real random variables, such that $\mathbb{E}(V(1-U)|UV)$ and $\mathbb{E}(V^2(1-U)^2|UV)$ are non-random then V has a gamma distribution and U has a beta distribution.

Keywords: Lukacs characterization; Conditional moments; Freeness; Free-Poisson distribution; Free-binomial distribution

1. Introduction

Characterizations of non-commutative variables and their distributions are a field which develops through non-commutative probability with results which parallel their classical counterparts. It is not completely well understood why the results mirror so much these from the classical setting since the nature of objects under study seems to be much different.

An example of such a result is the Bernstein characterization of the normal law of independent random variables X and Y by independence of $X+Y$ and $X-Y$ in classical probability [3] (see

* Corresponding author.

E-mail addresses: k.szpojankowski@mini.pw.edu.pl (K. Szpojankowski), j.wesolowski@mini.pw.edu.pl (J. Wesołowski).

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.09.015

also [21]), and a characterization of non-commutative semicircular variables $\mathbb{X}$ and $\mathbb{Y}$ which are free and such that $\mathbb{X}+\mathbb{Y}$ and $\mathbb{X}-\mathbb{Y}$ are free by Nica in [30]. Similarly, the classical characterization of the normal law by independence of the mean $\bar{X}=\frac{1}{n}\sum_{i=1}^n X_i$, and empirical variance $S^2=\frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X})^2$, where $(X_i)_{i=1,\dots,n}$ are independent, identically distributed real random variables from [22] is paralleled by a non-commutative characterization of the Wigner law exploiting freeness of $\bar{\mathbb{X}}=\frac{1}{n}\sum_{i=1}^n \mathbb{X}_i$ and $\mathbb{S}^2=\frac{1}{n-1}\sum_{i=1}^n (\mathbb{X}_i-\bar{\mathbb{X}})^2$ built on free identically distributed non-commutative random variables $(\mathbb{X}_i)_{i=1,\dots,n}$ – see [20].

In this paper we are concerned with the celebrated Lukacs characterization of the gamma distribution [28]. It says that if X and Y are positive, non-degenerate and independent random variables and such that

$$U=\frac{X}{X+Y} \quad \text{and} \quad V=X+Y \tag{1}$$

are independent then X and Y have gamma distributions, $G(p,a)$ and $G(q,a)$. Here by the gamma distribution $G(r,c)$, $r,c>0$, we understand the probability distribution with density

$$f(x)=\frac{c^r}{\Gamma(r)}x^{r-1}e^{-cx}I_{(0,\infty)}(x).$$

The direct result: If $X\sim G(p,a)$ and $Y\sim G(q,a)$ are independent then U and V, defined through (1), are independent; is rather simple. It suffices just to compute the Jacobian of the bijective transformation $(0,\infty)^2\ni(x,y)\mapsto(\frac{x}{x+y},x+y)\in(0,1)\times(0,\infty)$ and to follow how the densities transform. Immediately it follows also that $V\sim G(p+q,a)$ and U is a beta random variable $B_I(p,q)$, which has the density

$$f(x)=\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1}I_{(0,1)}(x).$$

The same computation while read backward proves the opposite implication: if U and V are independent, $U\sim G(p+q,a)$ and $V\sim B_I(p,q)$ then $X=UV$ and $Y=(1-U)V$ are independent, $X\sim G(p,a)$ and $Y\sim G(q,a)$.

For random matrices the role of the gamma law is taken over by Wishart distribution defined, e.g. on the cone $\mathcal{V}_+$ of non-negative definite real $n\times n$ symmetric matrices by the Laplace transform $L(\mathbf{s})=(\frac{\det\mathbf{a}}{\det(\mathbf{a}+\mathbf{s})})^p$ for positive definite $\mathbf{a}$ and $p\in\{0,\frac{1}{2},\frac{2}{2},\dots,\frac{n-1}{2}\}\cup(\frac{n-1}{2},\infty)$, and for $\mathbf{s}$ such that $\mathbf{a}+\mathbf{s}$ is positive definite. If $p>\frac{n-1}{2}$ then Wishart distribution has density with respect to the Lebesgue measure on $\mathcal{V}_+$ of the form

$$f(\mathbf{x})\propto(\det\mathbf{x})^{p-\frac{n+1}{2}}e^{-\operatorname{Tr}\mathbf{ax}}I_{\mathcal{V}_+}(\mathbf{x}).$$

Matrix variate beta distribution, in the case of real $n\times n$ matrices, is a probability distribution on the set $\mathcal{D}=\{\mathbf{x}\in\mathcal{V}_+\colon \mathbf{I}-\mathbf{x}\in\mathcal{V}_+\}$ defined by the density

$$g(\mathbf{x})\propto(\det\mathbf{x})^{p-1}\big(\det(\mathbf{I}-\mathbf{x})\big)^{q-1},$$

where the parameters $p,q\in(\frac{n-1}{2},\infty)$.