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Trudinger–Moser inequality with remainder terms ☆☆

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Abstract

The paper gives the following improvement of the Trudinger–Moser inequality:

$$\sup_{\int_\Omega |\nabla u|^2\,\mathrm{d}x - \psi(u) \leqslant 1,\; u \in C_0^\infty(\Omega)} \int_\Omega e^{4\pi u^2}\,\mathrm{d}x < \infty, \quad \Omega \Subset \mathbb{R}^2, \tag{0.1}$$

related to the Hardy–Sobolev–Mazya inequality in higher dimensions. We show (0.1) with $\psi(u) = \int_\Omega V(x)u^2\,\mathrm{d}x$ for a class of $V > 0$ that includes

$$V(r) = \frac{1}{4r^2(\log\frac{1}{r})^2 \max\{\sqrt{\log\frac{1}{r}}, 1\}},$$

which refines two previously known cases of (0.1) proved by Adimurthi and Druet [2] and by Wang and Ye [23]. In addition, we verify (0.1) for $\psi(u) = \lambda\|u\|_p^2$, as well as give an analogous improvement for the Onofri–Beckner inequality for the unit disk (Beckner [6]).

Keywords: Trudinger–Moser inequality; Borderline Sobolev imbeddings; Singular elliptic operators; Remainder terms; Spectral gap; Virtual bound state; Hardy–Sobolev–Mazya inequality

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1. Introduction

The Trudinger–Moser inequality [24,17,19,22,14]

$$\sup_{\int_\Omega |\nabla u|^2\,\mathrm{d}x \leqslant 1,\, u\in C_0^\infty(\Omega)} \int_\Omega e^{4\pi u^2}\,\mathrm{d}x < \infty, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, is an analog of the limiting Sobolev inequality in $\mathbb{R}^N$ with $N \geqslant 3$:

$$\sup_{\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x \leqslant 1,\, u\in C_0^\infty(\mathbb{R}^N)} \int |u|^{2^*}\,\mathrm{d}x < \infty, \quad 2^* = \frac{2N}{N-2}. \tag{1.2}$$

We recall that restriction of inequalities involving the gradient norm to bounded domains is of essence when $N = 2$, since the completion of $C_0^\infty(\mathbb{R}^2)$ in the gradient norm is not a function space, and, moreover, since $\int_B |\nabla u|^2\,\mathrm{d}x$ on the unit disk $B \subset \mathbb{R}^2$ coincides with the quadratic form of the Laplace–Beltrami operator on the hyperbolic plane (a *complete non-compact* Riemannian manifold) when expressed in the coordinates of the Poincaré disk.

Both limiting Trudinger–Moser and Sobolev inequalities are optimal in the sense that they are false for any nonlinearity that grows as $s \to \infty$ faster than $e^{4\pi s^2}$ resp. $|s|^{2^*}$. Inequality (1.2) is also false if the nonlinearity $|u|^{2^*}$ is multiplied by an unbounded radial monotone function, although (1.1) on the unit disk holds also when the integrand is replaced by $\frac{e^{4\pi u^2}-1}{(1-r)^2}$ [3,10].

This paper studies another refinement of (1.1), whose analogy in the case $N \geqslant 3$ is the Mazya's refinement of (1.2), known as Hardy–Sobolev–Mazya inequality [13]:

$$\sup_{\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x - \int_{\mathbb{R}^N} V_m(x)u^2\,\mathrm{d}x \leqslant 1,\, u\in C_0^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} |u|^{2^*}\,\mathrm{d}x < \infty, \tag{1.3}$$

where

$$V_m(x) = \left(\frac{m-2}{2}\right)^2 \frac{1}{|x_1 + \cdots + x_m|^2}, \quad m = 1, \ldots, N-1.$$

It is false when $m = N$, and similarly, inequality (0.1) does not hold with $\psi(u) = \int_B V(|x|)u^2\,\mathrm{d}x$, if V is the two-dimensional counterpart of the Hardy's radial potential, the Leray's potential

$$V_{\mathrm{Leray}}(r) = \frac{1}{4r^2(\log\frac{1}{r})^2}.$$

When $\psi(u) = \int_\Omega V(x)u^2\,\mathrm{d}x$, inequality (0.1) has been already established for two specific potentials V. In one case, proved by Adimurthi and Druet [2], $V(x) = \lambda < \lambda_1$, and λ_1 is the first eigenvalue of the Dirichlet Laplacian in Ω. Note only that the inequality stated as a main result in [2] is formally weaker, but it immediately implies (0.1) with $V(x) = \lambda < \lambda_1$ via an elementary argument. It was conjectured by Adimurthi [1] that the inequality remains valid whenever one replaces $\int_\Omega \lambda u^2\,\mathrm{d}x$ with a general weakly continuous functional ψ, as long as $\|\nabla u\|_2^2 - \psi(u) > 0$