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Polyhedrality in pieces ☆

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Abstract

The aim of this paper is to present two tools, Theorems 4 and 7, that make the task of finding equivalent polyhedral norms on certain Banach spaces easier and more transparent. The hypotheses of both tools are based on countable decompositions, either of the unit sphere S_X or of certain subsets of the dual ball B_{X^*} of a given Banach space X. The sufficient conditions of Theorem 4 are shown to be necessary in the separable case. Using Theorem 7, we can unify two known results regarding the polyhedral renorming of certain $C(K)$ spaces, and spaces having an (uncountable) unconditional basis. New examples of spaces having equivalent polyhedral norms are given in the final section.

Keywords: Polyhedral norms and renormings; Boundaries; Polytopes; Countable covers

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1. Introduction

Different notions of polyhedrality in infinite-dimensional spaces were considered in [9], as well as the relations between them. In this paper, we consider the original notion of polyhedrality given by Klee [16]: a Banach space X is said to be *polyhedral* when the unit balls of its finite-dimensional subspaces are polytopes.

We are interested in finding conditions that allow us to replace the norm on a given Banach space with an equivalent polyhedral norm. When a Banach space can be renormed in this way, it is called *isomorphically polyhedral*. Let us recall the main tool used in this endeavour.

Definition 1 *(The* $(*)$ *property).* Let X be a Banach space. We say that a set $F \subset X^*$ has property $(*)$ if, for every w^*-limit point g of F (i.e. any w^*-neighbourhood of g contains infinitely many points of F), we have $g(x) < 1$ whenever $\sup\{f(x)\colon f \in F\} = 1$.

It is known (see e.g. [7, Proposition 6.11]) that X is polyhedral whenever there is a 1-norming subset $B \subset B_{X^*}$ of the dual unit ball having $(*)$. Most of the notions considered in [9] imply that there exists one of these 1-norming sets. Property $(*)$ was introduced in [10], in the particular case of the set of extreme points $\operatorname{ext}(B_{X^*})$. In [10], the authors proved that if $\operatorname{ext}(B_{X^*})$ has $(*)$ then X is polyhedral. Moreover, if X is a Lindenstrauss space (that is, an isometric predual of some $L^1(\mu)$), then this condition is necessary too.

Recall that a subset $B \subset B_{X^*}$ is called a *boundary* if, for every $x \in S_X$, there exists $f_x \in B$ such that $f_x(x) = 1$. For any space, the unit sphere S_{X^*} and the extreme points $\operatorname{ext}(B_{X^*})$ of the unit ball are boundaries, by the Hahn–Banach and Krein–Milman Theorems, respectively. If B is a 1-norming subset of the dual unit ball having $(*)$, then it is automatically a boundary. The set of boundaries of a space is highly sensitive to changes to the norm. It is known that if X is a separable Banach space, then the following three statements are equivalent: X is isomorphically polyhedral, X admits an equivalent norm that supports a *countable* boundary, and X admits an equivalent norm that supports a countable boundary having $(*)$ [4].

However, finding renormings that support boundaries having $(*)$, countable or otherwise, can be quite difficult, even for concrete separable spaces. In this paper we introduce two tools, Theorems 4 and 7, that can be used, often in conjunction, to make the task of finding polyhedral renormings easier and more transparent. The hypotheses of both tools are based on countable decompositions, either of the unit sphere S_X or of certain subsets of the dual ball B_{X^*} (or both, in the case of Theorem 4). We believe that these hypotheses are easier to verify in many concrete cases. We mention also that the hypotheses of Theorem 4 are also necessary in the case of separable Banach spaces (see Proposition 10).

We introduce Theorems 4 and 7 in this section, together with some of their corollaries. The proofs of these theorems reside in subsequent sections. Some applications and examples are given in the final section.

Before presenting Theorem 4, we need to generalize the notion of boundary.

Definition 2 *(Relative boundaries).* Let X be a Banach space. We shall say that a set $F \subset X^*$ is a *relative boundary* if, whenever $x \in X$ satisfies $\sup\{f(x)\colon f \in F\} = 1$, then there exists $f_x \in F$ such that $f_x(x) = 1$.

Relative boundaries are sometimes called *James boundaries* in the literature. A different generalization of boundary for pieces of a space can be found in [6].