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Chern–Simons limit of the standing wave solutions for the Schrödinger equations coupled with a neutral scalar field

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Abstract

We show the existence of standing wave solutions to the Schrödinger equation coupled with a neutral scalar field. We also verify the Chern–Simons limit for these solutions. More precisely we prove that solutions to Eqs. (1.3)–(1.4) converge to the unique positive radially symmetric solution of the nonlinear Schrödinger equation (1.6) as the coupling constant q goes to infinity.

Keywords: Schrödinger; Chern–Simons limit; Standing waves; Variational methods

1. Introduction

In this paper, we study the following system in $\mathbb{R}^2$:

$$\begin{aligned}
&\left(i\partial_t + \frac{1}{2m}\Delta\right)\psi - \frac{q}{4m^2}|\psi|^2\psi - \left(1 + \frac{\kappa q}{2m}\right)N\psi = 0,\\
&\left(\partial_{tt} - \Delta + \kappa^2 q^2\right)N + q\left(1 + \frac{\kappa q}{2m}\right)|\psi|^2 = 0.
\end{aligned} \tag{1.1}$$

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Here, $\psi : \mathbb{R}^{1,2} \to \mathbb{C}$ represents a wave function of the underlying particle and $N : \mathbb{R}^{1,2} \to \mathbb{R}$ is a neutral scalar field. The parameters $m, \kappa, q > 0$ represent the mass of the particle, the Chern–Simons coupling constant and the Maxwell coupling constant respectively. One can derive the system (1.1) from a non-relativistic Abelian Maxwell–Chern–Simons model in [8], from which the neutral scalar field N comes. In Section 2, we will derive (1.1) from the Lagrangian level.

We are interested in standing wave solutions of the form

$$\psi(t,x) = e^{i\omega t}u(x), \qquad N(t,x) = N(x), \tag{1.2}$$

where $\omega \in \mathbb{R}$. Then, we can rewrite (1.1) as

$$-\frac{1}{2m}\Delta u + \frac{q}{4m^2}|u|^2 u + \left(1 + \frac{\kappa q}{2m}\right)Nu + \omega u = 0, \tag{1.3}$$

$$\left(-\Delta + \kappa^2 q^2\right)N + q\left(1 + \frac{\kappa q}{2m}\right)|u|^2 = 0. \tag{1.4}$$

Our first result is concerned with the existence of solutions.

Theorem 1.1. *For any given positive parameters m, κ, q and ω, there exists a solution $(u, N) \in \bigcap_{k=0}^{\infty}(H_r^k(\mathbb{R}^2) \times H_r^k(\mathbb{R}^2))$ of the system (1.3)–(1.4). Moreover, $u > 0$ and $N < 0$ on $\mathbb{R}^2$.*

We mean, by $H_r^k(\mathbb{R}^2)$, the set of radial functions in the standard Sobolev space $H^k(\mathbb{R}^2)$. We prove Theorem 1.1 by variational method. Solving (1.4) for N, one may regard N as a function of u. Then, we find a suitable functional defined by u only and apply mountain pass theorem.

One of the important issues in the Maxwell–Chern–Simons theories is the verification of the Chern–Simons limit. In (1.1), the constants q^{-1} and κ represent the strength of the Maxwell interaction and the Chern–Simons interaction in the underlying system, respectively. The question is what happens if the Chern–Simons interaction is dominant over the Maxwell interaction. This corresponds to the problem of studying the asymptotic behavior of solutions to (1.1) as $q \to \infty$. In fact, one can formally derive that for fixed $\kappa > 0$ and $m > 0$, if (ψ_q, N_q) is a solution of $(1.1)_q$, then $(\psi_q, N_q) \to (\psi_\infty, N_\infty)$ as $q \to \infty$, where $N_\infty = -|\psi_\infty|^2/(2m\kappa)$ and ψ_∞ is a solutions of the following nonlinear Schrödinger equation

$$\left(i\partial_t + \frac{1}{2m}\Delta\right)\psi + \frac{1}{m\kappa}|\psi|^2\psi = 0. \tag{1.5}$$

The problem of verifying this limit rigorously is called the Chern–Simons limit. There have been some efforts for proving the Chern–Simons limit for the steady state solutions [9–11,19,20]. In this paper, we prove the Chern–Simons limit for the standing wave solutions obtained in Theorem 1.1. The equation for the standing wave to (1.5) is

$$-\frac{1}{2m}\Delta u + \omega u - \frac{1}{m\kappa}|u|^2 u = 0. \tag{1.6}$$

Concerning the Chern–Simons limit, we prove the following theorem.