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Complex time evolution in geometric quantization and generalized coherent state transforms

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Abstract

For the cotangent bundle T^*K of a compact Lie group K, we study the complex-time evolution of the vertical tangent bundle and the associated geometric quantization Hilbert space $L^2(K)$ under an infinite-dimensional family of Hamiltonian flows. For each such flow, we construct a generalized coherent state transform (CST), which is a unitary isomorphism between $L^2(K)$ and a certain weighted L^2-space of holomorphic functions. For a particular set of choices, we show that this isomorphism is naturally decomposed as a product of a Heisenberg-type evolution (for complex time $-\tau$) within $L^2(K)$, followed by a polarization-changing geometric-quantization evolution (for complex time $+\tau$). In this case, our construction yields the usual generalized Segal–Bargmann transform of Hall. We show that the infinite-dimensional family of Hamiltonian flows can also be understood in terms of Thiemann's "complexifier" method (which generalizes the construction of adapted complex structures).

Keywords: Geometric quantization; Coherent state transform

Contents

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1. Introduction

In [12], Hall initiated the study of the relationship between the coherent state transform (CST, also known as the generalized Segal–Bargmann transform for compact Lie groups) and geometric quantization. Recall that the geometric quantization (Hilbert space) of a symplectic manifold is the subspace of sections of a certain line bundle on the manifold, called the prequantum line bundle, which are covariantly constant along some choice of polarization.[1] For example, if the manifold is Kähler, one can take the $(1,0)$-tangent bundle for the polarization, and the geometric quantization is the space of square-integrable holomorphic sections of the prequantum line bundle. If the manifold is a cotangent bundle, one can take the complexified vertical tangent bundle for the polarization, and so long as half-forms are included, the geometric quantization is the space of square-integrable functions on the base manifold. The complexification of a compact Lie group K is diffeomorphic to the cotangent bundle of K, and as such it has both of these structures. Hall showed that the CST can be understood in terms of geometric quantization as a unitary isomorphism between the vertically polarized Hilbert space $L^2(K)$ and the Hilbert space for the standard Kähler polarization.

The problem of choice of polarization is a fundamental issue in geometric quantization. In good cases, one might hope that the quantization is independent of this choice, but it turns out that such a hope is simply too optimistic. For example, each almost complex structure on a symplectic manifold gives rise to an almost-Kähler quantum Hilbert space, and one can attempt to compare these Hilbert spaces by forming a Hilbert bundle over the neighborhood of a point in the space of almost complex structures. This bundle has a natural connection (generalizing the connections of Axelrod, Della Pietra and Witten [1] and Hitchin [16]) which is given by projecting the trivial connection in the trivial bundle whose fiber is the space of all square-integrable sections of the prequantum line bundle onto the almost-holomorphic subbundle. In the case that the symplectic manifold is a symplectic vector space and one restricts to translation invariant complex structures, this connection is known to be projectively flat [1,21]. On the other hand, if one considers the full family of almost complex structures, Foth and Uribe have shown that the connection is never projectively flat, even semiclassically [6]. The lesson here is that one can expect projective flatness only for certain restricted families of complex structures.

[1] In geometric quantization, a *polarization* is a complex involutive Lagrangian distribution.