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An approximation to Wiener measure and quantization of the Hamiltonian on manifolds with non-positive sectional curvature ☆

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Abstract

This paper gives a rigorous interpretation of a Feynman path integral on a Riemannian manifold M with non-positive sectional curvature. An L^2 Riemannian metric $G_{\mathcal{P}}$ is given on the space of piecewise geodesic paths $H_{\mathcal{P}}(M)$ adapted to the partition $\mathcal{P}$ of $[0,1]$, whence a finite-dimensional approximation of Wiener measure is developed. It is proved that, as $\mathrm{mesh}(\mathcal{P}) \to 0$, the approximate Wiener measure converges in an L^1 sense to the measure $\exp\{-\frac{2+\sqrt{3}}{20\sqrt{3}}\int_0^1 \mathrm{Scal}(\sigma(s))\,ds\}\,d\nu(\sigma)$ on the Wiener space $W(M)$ with Wiener measure ν. This gives a possible prescription for the path integral representation of the quantized Hamiltonian, as well as yielding such a result for the natural geometric approximation schemes originating in Andersson and Driver (1999) [3] and followed by Lim (2007) [34].

Keywords: Path integrals; Finite-dimensional approximations; Wiener measure; Infinite-dimensional analysis

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1. Introduction

Let M be a d-dimensional Riemannian manifold with metric g, fixed point $o \in M$, and Levi-Civita covariant derivative ∇. For the remainder of this paper we will assume that the curvature and its derivative are bounded on M. We will eventually also require that the sectional curvature of M is non-positive, making sure to mention when we impose this restriction.

The Wiener space of M consists of the continuous paths starting at o and parameterized on $[0, 1]$,

$$W(M) = \{\sigma \in C([0,1] \to M): \sigma(0) = o\}. \tag{1.1}$$

The Wiener measure associated to M is the unique probability measure ν on $W(M)$ such that,

$$\int_{W(M)} f(\sigma)\, d\nu(\sigma) = \int_{M^n} F(x_1, \ldots, x_n) \prod_{i=1}^{n} P_i(dx_i) \tag{1.2}$$

whenever f has the form $f(\sigma) = F(\sigma(s_1), \ldots, \sigma(s_n))$ where $\mathcal{P} = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$ is a partition of $[0, 1]$ and F is a bounded and measurable function. The measures $P_i(dx_i)$ are defined as $P_i(dx_i) := p_{\Delta_i s}(x_{i-1}, x_i)\, dx_i$, where $p_s(x, y)$ denotes the fundamental solution to the heat equation on M, $\Delta_i s = s_i - s_{i-1}$, and dx_i is the Riemannian volume form on M.

The purpose of this paper is to give a rigorous interpretation of a heuristic path integral on M having the form,

$$\frac{1}{Z} \int_{W(M)} f(\sigma(1)) \exp\left\{ \int_0^1 \left(-\frac{1}{2} \|\sigma'(s)\|^2 + V(s) \right) ds \right\} \mathcal{D}\sigma \tag{1.3}$$

via a finite-dimensional approximation to Wiener measure. The "derivation" of Eq. (1.3) follows from an application of Trotter's product formula and a limiting argument from which Z arises as a "normalization" constant that can either be interpreted as 0 or ∞, and $\mathcal{D}\sigma$ is an infinite-dimensional Lebesgue type measure which, in truth, does not exist. Moreover, V is a potential and $-\frac{1}{2}\|\sigma'(s)\|^2 + V(s)$ yields an energy term which is problematic since the weight of the space $W(M)$ lands on nowhere differentiable paths.

In spite of the need to give a rigorous interpretation, heuristic path integrals such as those in Eq. (1.3) have proven themselves useful and arise often in physics literature. Particularly, one