Journal of Functional Analysis 267 (2014) 660–677

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Backward stochastic differential equations associated with the vorticity equations

A.B. Cruzeiro [a,b,*], Z.M. Qian [c]

[a] *Grupo Física-Matemática, Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal*
[b] *Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*
[c] *Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, England, United Kingdom*

A R T I C L E   I N F O

Article history:
Received 3 April 2013
Accepted 9 May 2014
Available online 27 May 2014
Communicated by B. Driver

Keywords:
Navier–Stokes
Vorticity equations
Backward SDE

A B S T R A C T

In this paper we derive a non-linear version of the Feynman–Kac formula for the solutions of the vorticity equation in dimension 2 with space periodic boundary conditions. We prove the existence (global in time) and uniqueness for a stochastic terminal value problem associated with the vorticity equation in dimension 2. A particular class of terminal values provide, via these probabilistic methods, solutions for the vorticity equation.

1. Introduction

The Feynman–Kac formula, in its original form derived from the idea of path integration in Feynman's PhD thesis (which is now available in a new print [8]), is a representation formula for solutions of Schrödinger's equations, and in the hand of Kac, is an explicit formula written in terms of functional integrals with respect to the Wiener measure, the law of Brownian motion.

* Corresponding author.

http://dx.doi.org/10.1016/j.jfa.2014.05.011

Bismut [3], Pardoux–Peng [11] and Peng [12], by utilizing Itô's lemma together with Itô's martingale representation, have obtained an interesting non-linear version of Feynman–Kac's formula for solutions of semi-linear parabolic equations in terms of backward stochastic differential equations (BSDE). The goal of the present paper is to derive a Feynman–Kac formula for solutions of the Navier–Stokes equations in the same spirit of Bismut and Pardoux–Peng [11], and to study the random terminal problem of the stochastic differential equations associated with the vorticity equations.

The main idea contained in [3,11] may be described as the following. Let $u(t,x) = (u^1(t,x), \cdots, u^m(t,x))$ be a smooth solution to the Cauchy initial value problem of the following system of semi-linear parabolic equations

$$\frac{\partial}{\partial t}u^i - \nu\Delta u^i + f^i(u, \nabla u) = 0, \qquad u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^d \tag{1.1}$$

where $i = 1, \cdots, m$, and $\nu > 0$ a constant. Let $B = (B^1, \cdots, B^d)$ be the standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $x \in \mathbb{R}^d$ and $T > 0$. Let us read the solution u along Brownian motion B. More explicitly, let $Y_t = u(T-t, \sqrt{2\nu}B_t + x)$ for $t \in [0,T]$ and $Z_t = \nabla u(T-t, \sqrt{2\nu}B_t + x)$, where ∇u is the linear operator from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined by $\nabla u(\cdot, x)v = \frac{d}{d\epsilon}|_{\epsilon=0} u(\cdot, x + \epsilon v), v \in \mathbb{R}^d$. Applying Itô's formula to u and B we obtain

$$Y_T - Y_t = \int_t^T f(Y_s, Z_s)ds + \sqrt{2\nu}\int_t^T Z_s \cdot dB_s, \quad Y_T = u_0(B_T). \tag{1.2}$$

In literature, (1.1) may be written in differential form

$$dY = f(Y,Z)dt + \sqrt{2\nu}Z \cdot dB, \quad Y_T = \xi, \tag{1.3}$$

where the arguments s, t, etc. are suppressed if no confusion may arise. The differential equation above is an example of backward stochastic differential equations, where the terminal value $Y_T = \xi$ is given. The function f appearing on the right hand side of (1.3) is called the (non-linear) *driver*.

Pardoux–Peng [11] made an important observation. If the non-linear driver f in BSDE (1.3) is globally Lipschitz continuous, then there is a unique adapted solution pair (Y,Z) satisfying (1.3) for a random terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, which is not necessary in the form of $u_0(B_T)$. The solution u and its gradient ∇u in turn can be represented in terms of (Y,Z). This representation may be considered as a non-linear extension of Feynman–Kac's formula to semi-linear parabolic equations.

More recently, Kobylanski [9], Delarue [7], Briand–Hu [4], Tevzadze [13], etc. have extended Pardoux–Peng's result to some BSDEs with non-linear drivers of quadratic growth. These papers however mainly deal with scalar BSDEs only, which corresponds to semi-linear scalar parabolic equations. It remains largely an open problem whether