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Irrationally elliptic closed characteristics on compact convex hypersurfaces in $\mathbf{R}^6$

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ABSTRACT

In this paper, let $\Sigma \subset \mathbf{R}^6$ be a compact convex hypersurface which carries exactly three geometrically distinct closed characteristics. We prove that at least two of them must be irrationally elliptic.

1. Introduction and main results

In this paper, let Σ be a fixed C^3 compact convex hypersurface in $\mathbf{R}^{2n}$, i.e., Σ is the boundary of a compact and strictly convex region U in $\mathbf{R}^{2n}$. We denote the set of

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all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We consider closed characteristics (τ, y) on Σ, which are solutions of the following problem

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\tau) = y(0), \end{cases} \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in $\mathbf{R}^n$, $\tau > 0$, $N_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, y) is *prime*, if τ is the minimal period of y. Two closed characteristics (τ, y) and (σ, z) are *geometrically distinct*, if $y(\mathbf{R}) \neq z(\mathbf{R})$. We denote by $\mathcal{J}(\Sigma)$ and $\widetilde{\mathcal{J}}(\Sigma)$ the set of all closed characteristics (τ, y) on Σ with τ being the minimal period of y and the set of all geometrically distinct ones respectively. Note that $\mathcal{J}(\Sigma) = \{\theta \cdot y \mid \theta \in S^1,\ y \text{ is prime}\}$, while $\widetilde{\mathcal{J}}(\Sigma) = \mathcal{J}(\Sigma)/S^1$, where the natural S^1-action is defined by $\theta \cdot y(t) = y(t + \tau\theta)$, $\forall \theta \in S^1$, $t \in \mathbf{R}$.

Let $j : \mathbf{R}^{2n} \to \mathbf{R}$ be the gauge function of Σ, i.e., $j(\lambda x) = \lambda$ for $x \in \Sigma$ and $\lambda \geq 0$, then $j \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^0(\mathbf{R}^{2n}, \mathbf{R})$ and $\Sigma = j^{-1}(1)$. Fix a constant $\alpha \in (1, 2)$ and define the Hamiltonian function $H_\alpha : \mathbf{R}^{2n} \to [0, +\infty)$ by

$$H_\alpha(x) = j(x)^\alpha, \quad \forall x \in \mathbf{R}^{2n}. \tag{1.2}$$

Then $H_\alpha \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. It is well known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

$$\begin{cases} \dot{y}(t) = JH'_\alpha(y(t)), \quad H_\alpha(y(t)) = 1, \ \forall t \in \mathbf{R}, \\ y(\tau) = y(0). \end{cases} \tag{1.3}$$

Denote by $\mathcal{J}(\Sigma, \alpha)$ the set of all solutions (τ, y) of (1.3) where τ is the minimal period of y and by $\widetilde{\mathcal{J}}(\Sigma, \alpha)$ the set of all geometrically distinct solutions of (1.3). As above, $\widetilde{\mathcal{J}}(\Sigma, \alpha)$ is obtained from $\mathcal{J}(\Sigma, \alpha)$ by dividing the natural S^1-action. Note that elements in $\mathcal{J}(\Sigma)$ and $\mathcal{J}(\Sigma, \alpha)$ are one to one correspondent to each other, similarly for $\widetilde{\mathcal{J}}(\Sigma)$ and $\widetilde{\mathcal{J}}(\Sigma, \alpha)$.

Let $(\tau, y) \in \mathcal{J}(\Sigma, \alpha)$. The fundamental solution $\gamma_y : [0, \tau] \to \mathrm{Sp}(2n)$ with $\gamma_y(0) = I_{2n}$ of the linearized Hamiltonian system

$$\dot{w}(t) = JH''_\alpha\big(y(t)\big)w(t), \quad \forall t \in \mathbf{R}, \tag{1.4}$$

is called the *associate symplectic path* of (τ, y). The eigenvalues of $\gamma_y(\tau)$ are called *Floquet multipliers* of (τ, y). By Proposition 1.6.13 of [7], the Floquet multipliers with their multiplicities of $(\tau, y) \in \mathcal{J}(\Sigma)$ do not depend on the particular choice of the Hamiltonian function in (1.3). For any $M \in \mathrm{Sp}(2n)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle