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Asymmetric Poincaré inequalities and solvability of capillarity problems ☆

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ABSTRACT

In this paper we first establish an asymmetric version of the Poincaré inequality in the space of bounded variation functions, and then, basically relying on this result, we discuss the existence, the non-existence and the multiplicity of bounded variation solutions of a class of capillarity problems with asymmetric perturbations.

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1. Introduction

The aim of this work is twofold: on the one hand we establish an asymmetric version of the Poincaré inequality in the space of bounded variation functions, on the other hand, based on this result, we prove the existence of bounded variation solutions of a class of capillarity problems with asymmetric perturbations.

Let Ω be a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$. The classical Poincaré inequality in $BV(\Omega)$ (see, e.g., [2, Theorem 3.44]) asserts that there exists a constant $c = c(\Omega) > 0$ such that every $v \in BV(\Omega)$, with $\int_\Omega v\,dx = 0$, satisfies

$$c\int_\Omega |v|\,dx \leqslant \int_\Omega |Dv|. \tag{1}$$

Here and in the sequel $\int_\Omega |Dv|$ denotes the total variation of v. The largest constant $c = c(\Omega)$ for which (1) holds is called the Poincaré constant and it is variationally characterized by

$$c = \min\left\{\int_\Omega |Dv|:\ v \in BV(\Omega),\ \int_\Omega v\,dx = 0,\ \int_\Omega |v|\,dx = 1\right\}.$$

Clearly, any minimizer yields the equality in (1).

The study of Poincaré and Sobolev–Poincaré inequalities, as well as of their variants and generalizations, is a very classical topic in functional analysis and it is still a field of very active research, in various different directions (see, e.g., [10,8,9,13,43] and the references contained therein).

In this paper we prove an asymmetric counterpart of the Poincaré inequality (1), where v^+ and v^- weigh differently, i.e., the ratio

$$r = \frac{\int_\Omega v^+\,dx}{\int_\Omega v^-\,dx}$$

is not necessarily 1. Namely, we show that for each $r > 0$ there exist constants $\mu = \mu(r,\Omega) > 0$ and $\nu = \nu(r,\Omega) > 0$, with $\nu = r\mu$, such that every $v \in BV(\Omega)$, with

$$\mu\int_\Omega v^+\,dx - \nu\int_\Omega v^-\,dx = 0,$$

satisfies

$$\mu\int_\Omega v^+\,dx + \nu\int_\Omega v^-\,dx \leqslant \int_\Omega |Dv|. \tag{2}$$