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Polynomials in operator space theory

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A R T I C L E   I N F O

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A B S T R A C T

The aim of this article is to start a metric theory of homogeneous polynomials in the category of operator spaces. For this purpose we take advantage of the basic fact that the space $P^m(E)$ of all m-homogeneous polynomials on a vector space E can be identified with the algebraic dual of the m-th symmetric tensor product $\otimes^{m,s}E$. Given an operator space V, we study several different types of completely bounded polynomials on V which form the operator space duals of $\otimes^{m,s}V$ endowed with related operator structures. Of special interest are what we call Haagerup, Kronecker, and Schur polynomials – polynomials associated with different types of matrix products.

1. Introduction

A basis-free abstract theory of polynomials on vector spaces was developed from 1909 onwards by Fréchet, Gâteaux, Michal and others, and independently also by Banach – at the same time as he invented the concept of normed spaces. Banach suggested the analysis of polynomials on normed spaces and had the intention of writing a book devoted to this non-linear part of his theory. Unfortunately he died in 1945 without realizing his project.

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The definition of polynomials on vector spaces, which is now commonly accepted as the most elegant one, is a definition via multilinear mappings: A mapping p between two vector spaces E and F is called m-homogeneous polynomial, $p \in P^m(E;F)$, if there exists an m-linear mapping $\varphi : E \times \cdots \times E \to F$, $\varphi \in L(^m E;F)$, such that $\varphi(x,\dots,x) = p(x)$ for all $x \in E$. Then there even is a unique symmetric m-linear mapping $\check{p}$ which on all diagonal elements $(x,\dots,x)$ equals $p(x)$. This is a consequence of the so-called polarization formula which defines $\check{p} : E \times \cdots \times E \to F$ through

$$\check{p}(x_1,\dots,x_m) = \frac{1}{m!2^m} \sum_{\delta_1,\dots,\delta_m \in \{-1,1\}} \delta_1 \cdots \delta_m p\Bigg(\sum_{k=1}^m \delta_k x_k\Bigg). \tag{1}$$

Symmetric m-linear mappings can be linearized by means of the m-fold symmetric tensor product. In order to understand this define the symmetric m-linear mapping

$$\vee : E \times \cdots \times E \to \otimes^m E, \quad (x_1,\dots,x_m) \mapsto \frac{1}{m!} \sum_{\eta \in \mathscr{S}_m} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(m)}$$

where the sum is taken over all possible permutations η on $\{1,\dots,m\}$ and $\otimes^m E$ stands for the m-fold tensor product of E. The m-fold symmetric tensor product $\otimes^{m,s} E$ then by definition is the linear span of the range of $\vee$. As a consequence, for every symmetric m-linear mapping $\varphi : E \times \cdots \times E \to F$ there is a unique linear mapping $\varphi^{L,s} : \otimes^{m,s} E \to F$ such that $\varphi^{L,s} \circ \vee = \varphi$. One ends up with the following algebraic identification:

$$\begin{aligned} P^m(E;F) &= L\big(\otimes^{m,s} E;F\big), \\ p &\longmapsto \check{p}^{L,s}. \end{aligned} \tag{2}$$

Let now E and F be two normed spaces. Then the continuous analog of the preceding equality is slightly more complicated. Let $\mathscr{P}^m(E;F)$ be the linear space of all continuous m-homogeneous polynomials which together with

$$\|p\| := \sup_{\|x\|_E \leqslant 1} \big\|p(x)\big\|_F$$

again forms a normed space (clearly, we have $\mathscr{P}^1(E;F) = \mathscr{L}(E;F)$, the space of all bounded linear operators). Inspired by Grothendieck's famous "Résumé de la théorie métrique des produits tensoriels topologiques" Ryan in [13] developed the concept of studying continuous m-homogeneous polynomials on normed spaces by analyzing symmetric tensor products. Define for $x \in \otimes^{m,s} E$ the projective symmetric tensor norm to be

$$\|x\|_{\pi_s} := \inf\Bigg\{\sum_j \|x_j\|^m \;\Bigg|\; x = \sum_j \otimes^m x_j\Bigg\}, \tag{3}$$