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Spectral gaps of Schrödinger operators and diffusion operators on abstract Wiener spaces

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ABSTRACT

In this paper we extend the spectral gap comparison theorem of Andrews and Clutterbuck (2011) [2] to the infinite dimensional setting. More precisely, we prove that the spectral gap of Schrödinger operator $-\mathcal{L}_* + V$ ($\mathcal{L}_*$ is the Ornstein–Uhlenbeck operator) on an abstract Wiener space is greater than that of the one-dimensional operator $-\frac{\mathrm{d}^2}{\mathrm{d}s^2} + s\frac{\mathrm{d}}{\mathrm{d}s} + \tilde{V}(s)$, provided that $\tilde{V}$ is a modulus of convexity for V. Similar result is established for the diffusion operator $-\mathcal{L}_* + \nabla F \cdot \nabla$.

1. Introduction

In this paper, we will compare the spectral gap of a self-adjoint Schrödinger operator on an abstract Wiener space with that on one-dimensional Gaussian space. This work is a natural extension of the famous *fundamental gap conjecture*, solved by Andrews and Clutterbuck [2] most recently, which gave an optimal lower bound of $\lambda_1 - \lambda_0$, the distance between the first two Dirichlet eigenvalues of a Schrödinger operator $-\Delta + V$

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on a bounded uniformly convex domain Ω with a weakly convex potential V. For the original conjecture with its literature, we refer to [2] and references therein.

First of all, we briefly recall Andrews and Clutterbuck's arguments. They introduced the notion of *modulus of convexity* for V which plays an important role in their proof. Namely, an even function $\tilde{V} \in C^1(-\frac{D}{2}, \frac{D}{2})$ ($D = \operatorname{diam}(\Omega)$) is called a modulus of convexity for V if

$$\big(\nabla V(x) - \nabla V(y)\big) \cdot \frac{x-y}{|x-y|} \geqslant 2\tilde{V}'\left(\frac{|x-y|}{2}\right), \quad \forall x \neq y \in \Omega.$$

Under this assumption, they estimated a *modulus of log-concavity* for the ground state ϕ_0 (i.e. the positive eigenfunction associated to λ_0):

$$\big(\nabla \log \phi_0(x) - \nabla \log \phi_0(y)\big) \cdot \frac{x-y}{|x-y|} \leqslant 2(\log \tilde{\phi}_0)'\left(\frac{|x-y|}{2}\right),$$

where $\tilde{\phi}_0$ is the ground state of operator $-\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \tilde{V}(s)$ with Dirichlet boundary condition. Moreover, let ϕ_1 be the eigenfunction associated to λ_1, it is known that $u_i = e^{-\lambda_i t}\phi_i$ ($i = 0, 1$) are two solutions to the Dirichlet heat equation $\frac{\partial u}{\partial t} = \Delta u - Vu$ on $\Omega \times [0, \infty)$, and the *ground transformation* $v = \frac{u_1}{u_0} = e^{-(\lambda_1 - \lambda_0)t} \cdot \frac{\phi_1}{\phi_0}$ solves the Neumann heat equation $\frac{\partial v}{\partial t} = \Delta u + 2\nabla \log u_0 \cdot \nabla v$. Due to that the drift field $\nabla \log u_0$ has a modulus of log-concavity as above, they derived a *modulus of continuity* for v

$$v(x,t) - v(y,t) \leqslant e^{-(\lambda_1 - \lambda_0)t} \operatorname{osc} \frac{\phi_1}{\phi_0} \leqslant 2Ce^{-(\tilde{\lambda}_1 - \tilde{\lambda}_0)t},$$

where $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ denote the first two Dirichlet eigenvalues of $-\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \tilde{V}(s)$. Consequently, letting $t \to \infty$ gives the comparison

$$\lambda_1 - \lambda_0 \geqslant \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

This is the beautiful strategy in [2] to solve the conjecture, which works for smooth potentials and compact domains.

Our purpose is to generalize these results to get quantitative estimates of spectral gaps and ground states for Schrödinger type operators in infinite dimensional settings. This kind of problem can be set in the context of mathematical spectral theory of quantum field models. Recall that, Simon and Hoegh-Krohn [12] presented a comprehensive study to the perturbation theory for Schrödinger type operators on abstract Boson Fock spaces. They showed the existence of discrete spectrum in some interval provided that the potential function satisfies certain exponential integrability (for example, see [12, Theorem 4.5]). In particular, for Schrödinger operators on loop spaces over compact connected Riemannian manifolds, F.-Z. Gong, M. Röckner and L.-M. Wu [6] proved the Poincaré inequality which thus solved L. Gross' conjecture on the existence of spectral