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Symmetric functions of two noncommuting variables

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ABSTRACT

We prove a noncommutative analogue of the fact that every symmetric analytic function of (z, w) in the bidisc $\mathbb{D}^2$ can be expressed as an analytic function of the variables $z+w$ and zw. We construct an analytic nc-map S from the biball to an infinite-dimensional nc-domain Ω with the property that, for every bounded symmetric function φ of two noncommuting variables that is analytic on the biball, there exists a bounded analytic nc-function Φ on Ω such that $\varphi = \Phi \circ S$. We also establish a realization formula for Φ, and hence for φ, in terms of operators on Hilbert space.

1. Introduction

Every symmetric polynomial in two commuting variables z and w can be written as a polynomial in the variables $z + w$ and zw; conversely every polynomial in $z + w$ and zw determines a symmetric polynomial in z and w. A similar assertion holds for symmetric analytic functions on symmetric domains in $\mathbb{C}^2$. For noncommuting variables, on the other hand, no such simple characterizations are valid. For example, the polynomial

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$$zwz + wzw$$

in noncommuting variables z, w *cannot* be written as $p(z+w, zw+wz)$ for any polynomial p; M. Wolf showed in 1936 [11] that there is no finite basis for the ring of symmetric noncommuting polynomials over $\mathbb{C}$. She gave noncommutative analogues of the elementary symmetric functions, but they are infinite in number.

In this paper we extend Wolf's results from polynomials to symmetric analytic functions in noncommuting variables within the framework of noncommutative analysis, as developed by J.L. Taylor [9] and many other authors, for example [2,3,5–7,10]. We prove noncommutative analogues of the following simple classical result.

Let $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$\pi(z, w) = (z + w, zw).$$

If $\varphi : \mathbb{D}^2 \to \mathbb{C}$ is analytic and symmetric in z and w then there exists a unique analytic function $\Phi : \pi(\mathbb{D}^2) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{D}^2 \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & \pi(\mathbb{D}^2) \arrow[dl, "\Phi"] \\ & \mathbb{C} & \end{tikzcd}$$

In this diagram the domain $\pi(\mathbb{D}^2)$ is two-dimensional, in consequence of the fact that there is a basis of the ring of symmetric polynomials consisting of two elements, $z + w$ and zw. Wolf's result implies that in any analogous statement for symmetric polynomials in two *noncommuting* variables, $\pi(\mathbb{D}^2)$ will have to be replaced by an infinite-dimensional domain. The same will necessarily be true for the larger class of symmetric holomorphic functions of two noncommuting variables.

We use the notions of *nc-functions* and *nc-maps* on *nc-domains*, briefly explained in Section 2. An example of an nc-domain is the *biball*

$$B^2 \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} B_n \times B_n,$$

where B_n denotes the open unit ball of the space $\mathcal{M}_n$ of $n \times n$ complex matrices. B^2 is the noncommutative analogue of the bidisc. It is a symmetric domain in the sense that if $(x^1, x^2) \in B^2$ then also $(x^2, x^1) \in B^2$. Another example of an nc-domain is the space

$$\mathcal{M}^{\infty} \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathcal{M}_n^{\infty}$$

of infinite sequences of $n \times n$ matrices, for any $n \geqslant 1$.

The following result is contained in Theorem 5.1 below. An nc-function φ on B^2 is said to be *symmetric* if $\varphi(x^1, x^2) = \varphi(x^2, x^1)$ for all $(x^1, x^2) \in B^2$.