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Compact lines and the Sobczyk property

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ABSTRACT

We show that Sobczyk's Theorem holds for a new class of Banach spaces, namely spaces of continuous functions on linearly ordered compacta.

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1. Introduction

The main result of this paper (Corollary 2.5) states that, for every compact line K, the Banach space $C(K)$ of continuous real-valued functions on K (endowed with the supremum norm) has the *Sobczyk property*, i.e., every isomorphic copy of c_0 in $C(K)$ is complemented. By a *compact line* we mean a linearly ordered set which is compact in the order topology (see [2, 1.7.4] for the definition and basic facts about the order topology and [2, 3.12.3] for the characterization of linear orders yielding a compact topology). Topological properties of compact lines and structural properties of their spaces of continuous functions have recently been studied in a series of articles [1,7,6,10].

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In his celebrated theorem [16], Sobczyk has proven that every separable Banach space has the property now named after him. Such result follows from the fact that c_0 is *separably injective*, i.e., given a separable Banach space X and a closed subspace Y of X, every bounded operator $T : Y \to c_0$ admits a bounded extension to X. When the latter condition holds for a closed subspace Y of a Banach space X, we say that Y has the *c_0-extension property* (briefly: c_0EP) in X. Moreover, a Banach space X is said to have the (resp., *separable*) *c_0-extension property* if every (resp., separable) closed subspace of X has the c_0EP in X. The c_0EP for Banach spaces has been studied by the authors in [1].

The main result of the present paper follows from our Theorem 2.4 which states that, if K is a compact line, then Banach subspaces of $C(K)$ with separable dual have the c_0EP in $C(K)$. The latter generalizes [1, Theorem 3.1], which implies that, for a compact line K, Banach subalgebras of $C(K)$ with separable dual have the c_0EP in $C(K)$ (though — see Remark 2.8 — the extension constant obtained in [1, Theorem 3.1] is smaller). Note that, although separable Banach subspaces of $C(K)$ always span separable Banach subalgebras, even a one-dimensional subspace of $C(K)$ can span a Banach subalgebra with nonseparable dual. (By the *Banach subalgebra spanned* by a subset of $C(K)$ we mean the smallest Banach subalgebra of $C(K)$ containing that set.) Thus Theorem 2.4 is much stronger than [1, Theorem 3.1]: the latter does not imply that $C(K)$ has the Sobczyk property.

Let us briefly review some known results stating that certain classes of Banach spaces have the Sobczyk property. We remark that their proofs can usually be adapted to establish the c_0EP or the separable c_0EP. For instance, a simple adaptation of Veech's proof [20] of Sobczyk's Theorem shows that weakly compactly generated (briefly: WCG) Banach spaces have the c_0EP (see also [1, Proposition 2.2] for details). Moltó's argument [13, Theorem 3] shows that, assuming a certain topological condition for the dual ball (B_{X^*}, w^*), one obtains that the Banach space X has the separable c_0EP. (Moltó's topological condition is satisfied, for instance, by all Corson compacta.) A Banach space X is said to satisfy the *separable complementation property* (briefly: SCP) if every separable subspace of X is contained in a separable complemented Banach subspace of X. Sobczyk's Theorem implies that all Banach spaces with the SCP have the separable c_0EP. In [19, Lemma, p. 494] it is shown that if K is a Valdivia compact space then $C(K)$ has the SCP. Since the separable c_0EP is hereditary to closed subspaces, it follows also that $C(K)$ has the separable c_0EP when K is a continuous image of a Valdivia compactum.

Recall that a compact Hausdorff space K is said to be *$\aleph_0$-monolithic* if every separable subspace of K is metrizable. It is easy to show (see [1, Corollary 2.7]) that if K is $\aleph_0$-monolithic then $C(K)$ has the separable c_0EP. It turns out that if K is a compact line then $C(K)$ has the separable c_0EP only in the trivial case when K is $\aleph_0$-monolithic (Theorem 2.2), though $C(K)$ always has the Sobczyk property.

The *double arrow space* $\mathrm{DA} = [0, 1] \times \{0, 1\}$ (endowed with the lexicographic order and order topology) is a separable nonmetrizable compact line and thus every compact line

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