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Independence properties in subalgebras of ultraproduct II_1 factors ☆

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ABSTRACT

Let M_n be a sequence of finite factors with $\dim(M_n) \to \infty$ and denote by $\mathbf{M} = \prod_\omega M_n$ their ultraproduct over a free ultrafilter ω. We prove that if $\mathbf{Q} \subset \mathbf{M}$ is either an ultraproduct $\mathbf{Q} = \prod_\omega Q_n$ of subalgebras $Q_n \subset M_n$, with $Q_n \not\prec_{M_n} Q_n' \cap M_n$, $\forall n$, or the centralizer $\mathbf{Q} = B' \cap \mathbf{M}$ of a separable amenable *-subalgebra $B \subset \mathbf{M}$, then for any separable subspace $X \subset \mathbf{M} \ominus (\mathbf{Q}' \cap \mathbf{M})$, there exists a diffuse abelian von Neumann subalgebra in $\mathbf{Q}$ which is *free independent* to X, relative to $\mathbf{Q}' \cap \mathbf{M}$. Some related independence properties for subalgebras in ultraproduct II_1 factors are also discussed.

0. Introduction

We continue in this paper the investigation of independence properties in subalgebras of ultraproduct II_1 factors, from [29,34]. The main result we prove along these lines is the following:

0.1. Theorem. *Let M_n be a sequence of finite factors with $\dim M_n \to \infty$ and denote by $\mathbf{M}$ the ultraproduct II_1 factor $\prod_\omega M_n$, over a free ultrafilter ω on $\mathbb{N}$. Let $\mathbf{Q} \subset \mathbf{M}$ be a von Neumann subalgebra satisfying one of the following:*

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(a) $\mathbf{Q} = \prod_\omega Q_n$, *for some von Neumann subalgebras* $Q_n \subset M_n$ *satisfying the condition* $Q_n \not\prec_{M_n} Q_n' \cap M_n$, $\forall n$ (*in the sense of* [32]);
(b) $\mathbf{Q} = B' \cap \mathbf{M}$, *for some separable amenable von Neumann subalgebra* $B \subset \mathbf{M}$.

Then given any separable subspace $X \subset \mathbf{M} \ominus (\mathbf{Q}' \cap \mathbf{M})$, *there exists a diffuse abelian von Neumann subalgebra* $A \subset \mathbf{Q}$ *such that* A *is free independent to* X, *relative to* $\mathbf{Q}' \cap \mathbf{M}$, *i.e.* $E_{\mathbf{Q}' \cap \mathbf{M}}(x_0 \prod_{i=1}^n a_i x_i) = 0$, *for all* $n \geqslant 1$, $x_0, x_n \in X \cup \{1\}$, $x_i \in X$, $1 \leqslant i \leqslant n-1$, $a_i \in A \ominus \mathbb{C}1$, $1 \leqslant i \leqslant n$.

Note that the particular case when $Q_n \subset M_n$ are II_1 factors with atomic relative commutant, for which one clearly has $Q_n \not\prec_{M_n} Q_n' \cap M_n$, recovers 2.1 in [29].

The conclusion in 0.1 above can alternatively be interpreted as follows: given any separable von Neumann subalgebra P of $\mathbf{M}$ that makes a commuting square with $\mathbf{Q}' \cap \mathbf{M}$ (in the sense of 1.2 in [25]; see Section 1.2 below for the definition) and we let $B_1 = P \cap (\mathbf{Q}' \cap \mathbf{M})$, there exists a separable von Neumann subalgebra $B_0 \subset \mathbf{Q}$, such that $P \vee B_0 \simeq P *_{B_1} (B_1 \overline{\otimes} B_0)$ (amalgamated free product of finite von Neumann algebras over a common subalgebra, see [36,27]). Since in the case (b) of 0.1 we have $\mathbf{Q}' \cap \mathbf{M} = B$ (see 2.1 below) and all embeddings into an ultraproduct II_1 factor $\mathbf{M}$ of an amenable separable von Neumann algebra B are conjugate by unitaries in $\mathbf{M}$, Theorem 0.1 shows in particular that if two separable finite von Neumann algebras N_1, N_2 containing copies of B are embeddable into $\mathbf{M}$, then $N_1 *_B N_2$ is embeddable into $\mathbf{M}$ as well. Note that the case B atomic of this result already appears in [29], while the case B arbitrary but with $\mathbf{M} = R^\omega$ was shown in [4]. More precisely, 0.1 implies the following strengthening of these results:

0.2. Corollary. *Let* $\mathbf{M} = \prod_\omega M_n$ *be an ultraproduct* II_1 *factor as in* 0.1. *Let* $N_i \subset \mathbf{M}$ *be separable finite von Neumann subalgebras with amenable von Neumann subalgebras* $B_i \subset N_i$, $i = 1, 2$, *such that* $(B_1, \tau_{|B_1}) \simeq (B_2, \tau_{|B_2})$. *Then there exists a unitary element* $u \in \mathbf{M}$ *so that* $uB_1u^* = B_2$ *and so that, after identifying* $B = B_1 \simeq B_2$ *this way, we have* $uN_1u^* \vee N_2 \simeq N_1 *_B N_2$.

To prove Theorem 0.1, we first construct unitaries $u \in \mathbf{Q}$ that are approximately n-independent with respect to given finite sets $X \perp \mathbf{Q}' \cap \mathbf{M}$. Taking larger and larger n, larger and larger finite sets X and better and better approximations, and combining with a diagonalization procedure, one can then get unitaries that are free independent to a given countable set, due to the ultraproduct framework.

The approximately independent unitary u is constructed by patching together incremental pieces of it, while controlling the trace of alternating words involving u and a given set X. This technique was initiated in [26], being then fully developed in [29], where it has been used to prove a particular case of 0.1(a). More recently, it has been used in [34] to establish existence of free independence in ultraproducts of maximal abelian *-subalgebras (abbreviated hereafter MASA) $A_n \subset M_n$ that are *singular* in the sense of [7] (i.e., any unitary element in M_n that normalizes A_n must lie in A_n),