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Characterization of composition operators with closed range for one-dimensional smooth symbols

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ABSTRACT

We give a full characterization of smooth symbols $\psi : \mathbb{R} \to \mathbb{R}$ for which the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$ has closed range. This generalizes in a special case the result of Kenessey and Wengenroth who gave such a characterization for smooth *injective* symbols $\psi : \mathbb{R} \to \mathbb{R}^d$.

1. Introduction

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$ be the corresponding composition operator. This paper deals with the question of when this operator has closed range, i.e., when the set $\operatorname{Im} C_\psi = \{F \circ \psi\colon\ F \in C^\infty(\mathbb{R})\}$ is closed in $C^\infty(\mathbb{R})$. Our main result reads as follows.

Theorem. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth. The operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $F \mapsto F \circ \psi$ has closed range if and only if ψ is a semiproper function which is either constant or satisfies both of the following conditions:*

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1. Every fiber over a boundary point of $\psi(\mathbb{R})$ contains a non-flat point.
2. Every fiber over an interior point of $\psi(\mathbb{R})$ contains either a non-flat non-extreme point or both a non-flat local minimum and a non-flat local maximum.

The sufficiency part of this theorem was proved by the author in [11].

Recall that by $C^\infty(\mathbb{R})$ we denote the space of real valued smooth functions on the real line, equipped with the usual topology of uniform convergence of functions and all their derivatives on compact sets. This topology is generated by the family of seminorms:

$$\{\|\cdot\|_{n,L}\colon\ n \in \mathbb{N},\ L \text{ compact}\},$$

where

$$\|f\|_{n,L} := \max_{x\in L}\max_{0\leqslant i\leqslant n}\left|f^{(i)}(x)\right|,$$

and with this topology $C^\infty(\mathbb{R})$ is a Fréchet space. A function $\psi : \mathbb{R} \to \mathbb{R}$ is called *semiproper* if for every compact set $K \subset \mathbb{R}$ there exists a compact set $L \subset \mathbb{R}$ such that $K \cap \psi(\mathbb{R}) \subset \psi(L)$. It is well known that closed range composition operators must have semiproper symbols (see [5, Proposition 1.4.1] without proof, for proof see [8, p. 2003]). It is easy to see that a continuous function $\psi : \mathbb{R} \to \mathbb{R}$ is semiproper if and only if $\psi(\mathbb{R})$ is closed. If $\psi^{(n)}(x) = 0$ for all $n \in \mathbb{N}$ then we say that ψ is *flat* at x. A set of the form $\psi^{-1}(\{b\})$ is called the *fiber* of ψ over b.

Probably the first result about composition operators goes back to Whitney. In 1943 in [12] he proved that every smooth and even function on $\mathbb{R}$ can be represented as a composition of a smooth function with the function $\psi(x) = x^2$. Because the subspace of even functions is closed in $C^\infty(\mathbb{R})$, this is equivalent to the statement that the composition operator $C_\psi : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ with the symbol $\psi(x) = x^2$ has closed range.

A remarkable progress was later made by Glaeser [7]. He proved that if $\psi : \mathbb{R}^m \to \mathbb{R}^n$ (where $n \leqslant m$) is a semiproper analytic function with a dense set of points where its Jacobian has rank equal to n, then the composition operator $C_\psi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^m)$ has closed range. Finally, Bierstone and Milman [3,4] and Bierstone, Milman, and Pawłucki [6] gave a complete characterization of composition operators on $C^\infty(\Omega)$ with closed range for all analytic symbols ψ.

In the general case, when ψ is only a smooth function, the situation is far more complicated. In 1998 Allan, Kakiko, O'Farrell and Watson gave a description of the closure of $\operatorname{Im} C_\psi$ for smooth injective symbols $\psi : \mathbb{R} \to \mathbb{R}^d$ (see [1,2]). Using their results, in 2011, Kenessey and Wengenroth made a first step towards the description of closed range composition operators for smooth symbols. They proved the following theorem.

Theorem. *(Kenessey, Wengenroth, 2011) Let $\psi : \mathbb{R} \to \mathbb{R}^d$ be a smooth injective function. The composition operator $C_\psi : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R})$ has closed range if and only if ψ is semiproper, has Whitney regular image, and no flat points.*