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On the sum of a narrow and a compact operators

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ABSTRACT

Our main technical tool is a principally new property of compact narrow operators which works for a domain space without an absolutely continuous norm. It is proved that for every Köthe F-space X and for every locally convex F-space Y the sum $T_1 + T_2$ of a narrow operator $T_1 : X \to Y$ and a compact narrow operator $T_2 : X \to Y$ is a narrow operator. This gives a positive answers to questions asked by M. Popov and B. Randrianantoanina [6, Problems 5.6 and 11.63].

1. Introduction

The notion of narrow operators was introduced by A. Plichko and M. Popov in [5]. This notion was considered as some development of the notion of a compact operator. Narrow operators were studied by many mathematicians (see [6]). The question of whether a sum of two narrow operators has to be narrowed cause a special interest in the investigation of properties of narrow operators. The study of this question was conducted in two directions. The first of them consists of results on the narrowness of a sum of two narrow operators. It was obtained in [5] that a sum of two narrow operators on L_1 is narrow. The question on the narrowness of a sum of two narrow operators defined on L_1 with

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any range Banach space was arose in this connection (see [2]). In [3] the notion of narrow operators defined on Köthe function spaces was extended to operators defined on vector lattices. Results obtained in [3] imply that a sum T_1+T_2 of two regular narrow operators $T_1, T_2 : X \to Y$ defined on an order continuous atomless Banach lattice X and with values in an order continuous Banach lattice Y is narrow.

The second direction of the investigations contains examples of two narrow operators the sum of which is non-narrow. It was proved in [5] that for any rearrangement invariant space X on $[0,1]$ with an unconditional basis every operator on X is a sum of two narrow operators. In [4] M. Popov and the author proved that for any Köthe Banach space X on $[0,1]$, there exist a Banach space Y and narrow operators $T_1, T_2 : X \to Y$ with a non-narrow sum $T = T_1+T_2$. This answers in the negative the question of V. Kadets and M. Popov from [2]. Moreover, an example of regular narrow operators $T_1, T_2 : L_p \to L_\infty$, where $p \in (1,\infty]$, with a non-narrow sum $T_1 + T_2$ was constructed in [4].

Furthermore, the narrowness of the sum of a narrow and a compact operators was investigated too. It was proved in [5] that the sum of a narrow and a compact operators defined on a symmetric Banach space on $[0,1]$ with an absolutely continuous norm is narrow. Since L_∞ is the classical example of a symmetric Banach space the norm of which is not absolutely continuous, the following questions are natural.

Problem 1.1. (See [6, Problem 5.6].) Is a sum of two narrow functionals on L_∞ narrow?

Problem 1.2. (See [6, Problem 11.63].) Is a sum of two narrow operators on L_∞, at least one of which is compact, narrow?

It is worth mentioning that a compact operator defined on L_∞ need not be narrow. Moreover, there is a non-narrow continuous linear functional [3]. Thus, instead of asking whether the sum of a narrow and a compact operators is narrow, one should ask whether the sum of a narrow and a compact narrow operators is narrow.

In this paper we obtain a principally new property of compact narrow operator which works for a domain space without an absolutely continuous norm. Using this property we give a positive answer to Problems 1.1 and 1.2. More precisely, we show that for every Köthe F-space X and for every locally convex F-space Y the sum of a narrow operator $T_1 : X \to Y$ and a compact narrow operator $T_2 : X \to Y$ is narrow.

2. Preliminaries

For topological vector spaces X and Y by $\mathcal{L}(X,Y)$ we denote the space of all linear continuous operators $T : X \to Y$.

Let (Ω, Σ, μ) be a finite atomless measure space, let Σ^+ be the set of all $A \in \Sigma$ with $\mu(A) > 0$ and let $L_0(\mu)$ be the linear space of all equivalence classes of Σ-measurable functions $x : \Omega \to \mathbb{K}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.