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Journal of Functional Analysis 266 (2014) 792–813

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Global regularity for higher order divergence elliptic and parabolic equations

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Received 3 February 2013; accepted 14 October 2013

Available online 5 November 2013

Communicated by C. De Lellis

Abstract

In this paper we obtain the global regularity estimates of the weak solutions in Sobolev spaces and Orlicz spaces for higher order elliptic and parabolic equations of divergence form with small BMO coefficients in the whole space. We only focus on the parabolic case while the corresponding result in the elliptic case can be obtained as a corollary.

Keywords: Regularity; Sobolev; Orlicz spaces; Higher order; Divergence; Elliptic; Parabolic; Small BMO; The whole space

1. Introduction

In this paper we are mainly concerned with global regularity estimates in Sobolev spaces and Orlicz spaces for weak solutions of the following higher order elliptic and parabolic problems in divergence form

$$\mathcal{M}u =: D^{\alpha}\left(A_{\alpha\beta}(x)D^{\beta}u\right) = D^{\alpha} f_{\alpha}, \quad x \in \mathbb{R}^{n}, \tag{1.1}$$

and

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0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.10.018

$$\mathcal{L}u =: u_t + (-1)^m D^\alpha\big(A_{\alpha\beta}(x,t)D^\beta u\big) = D^\alpha f_\alpha, \quad (x,t) \in \mathbb{R}^n_T =: \mathbb{R}^n \times (0,T), \tag{1.2}$$

$$u(x,0) = 0, \tag{1.3}$$

where T is a positive fixed time, α, β are multi-indices with $|\alpha| = |\beta| = m$, m is a positive integer, $\mathbf{f} = \{f_\alpha\colon\ |\alpha| = m\}$ is a given function, and the tensor coefficients $A_{\alpha\beta}$ satisfy

$$\sup_{\alpha,\beta} \|A_{\alpha\beta}\|_{L^\infty} \leqslant \Lambda_1 \tag{1.4}$$

and

$$\Lambda_2^{-1}|\xi|^2 \leqslant A_{\alpha\beta}\xi^\alpha \cdot \xi^\beta \leqslant \Lambda_2|\xi|^2 \tag{1.5}$$

for all $x \in \mathbb{R}^n$, all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, all tensor matrices $\xi = \{\xi^\alpha\colon\ |\alpha| = m\}$ and two positive constants Λ_1, Λ_2. As usual, we use summation convention over repeated indices.

Especially when $m = 1$, (1.1) and (1.2) are reduced to the following second-order elliptic and parabolic problems

$$\operatorname{div}\big(A(x)\nabla u\big) = \operatorname{div}\mathbf{f} \tag{1.6}$$

and

$$u_t - \operatorname{div}\big(A(x,t)\nabla u\big) = \operatorname{div}\mathbf{f}. \tag{1.7}$$

L^p-type regularity is the fundamental theory for elliptic and parabolic equations, which plays an important role in the theory of partial differential equations, and is the basis for the existence and uniqueness of solutions. Many authors [7,8,11–13] have studied L^p-regularity theory of weak solutions for (1.6)–(1.7) with different assumptions on the coefficients and domains. Furthermore, there have been wide research activities [1,2,9,23,22,24,25] on the study on L^p-type regularity theory of weak solutions for the nonlinear second-order elliptic and parabolic equations. Recently, some authors [14,19–21] obtained the global L^p-type regularity for (1.6)–(1.7) since for issues related to stochastic processes it is enough to consider the corresponding PDEs in the whole space.

However, there are not many corresponding regularity results for the higher order elliptic and parabolic equations (for instance, see [15,17,26–31]). When $a_\nu \in L^\infty \cap VMO$, Palagachev and Softova [28,29] have studied local L^p-type regularity for the following nondivergence higher-order elliptic and parabolic equations

$$a_\nu(x)D^\nu u = f(x) \quad \text{and} \quad u_t - a_\nu(x,t)D^\nu u = f(x,t) \quad \text{for } |\nu| = 2m.$$

Moreover, Haller-Dintelmann, Heck and Hieber [17] extended the corresponding local result in [28] to the global case for the higher-order parabolic equation

$$u_t - a_\nu(x,t)D^\nu u = f(x,t) \quad \text{for } 0 \leqslant |\nu| \leqslant 2m, \tag{1.8}$$

when the leading coefficients a_ν for $|\nu| = 2m$ belong to the class $L^\infty \cap VMO$. Furthermore, Dong and Kim [15] have proved the global L^p-type regularity of problems (1.1), (1.2), (1.8) and

$$a_\nu(x)D^\nu u = f(x) \quad \text{for } 0 \leqslant |\nu| \leqslant 2m \tag{1.9}$$