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Convergence of densities of some functionals of Gaussian processes

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Abstract

The aim of this paper is to establish the uniform convergence of the densities of a sequence of random variables, which are functionals of an underlying Gaussian process, to a normal density. Precise estimates for the uniform distance are derived by using the techniques of Malliavin calculus, combined with Stein's method for normal approximation. We need to assume some non-degeneracy conditions. First, the study is focused on random variables in a fixed Wiener chaos, and later, the results are extended to the uniform convergence of the derivatives of the densities and to the case of random vectors in some fixed chaos, which are uniformly non-degenerate in the sense of Malliavin calculus. Explicit upper bounds for the uniform norm are obtained for random variables in the second Wiener chaos, and an application to the convergence of densities of the least square estimator for the drift parameter in Ornstein–Uhlenbeck processes is discussed.
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Keywords: Multiple Wiener–Itô integrals; Wiener chaos; Malliavin calculus; Integration by parts; Stein's method; Convergence of densities; Ornstein–Uhlenbeck process; Least squares estimator; Small deviation

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1. Introduction

There has been a recent interest in studying normal approximations for sequences of multiple stochastic integrals. Consider a sequence of multiple stochastic integrals of order $q \geqslant 2$, $F_n = I_q(f_n)$, with variance $\sigma^2 > 0$, with respect to an isonormal Gaussian process $X = \{X(h), h \in \mathfrak{H}\}$ associated with a Hilbert space $\mathfrak{H}$. It was proved by Nualart and Peccati [24] and Nualart and Ortiz-Latorre [23] that F_n converges in distribution to the normal law $N(0, \sigma^2)$ as $n \to \infty$ if and only if one of the following three equivalent conditions holds:

(i) $\lim_{n\to\infty} E[F_n^4] = 3\sigma^4$ (convergence of the fourth moments).
(ii) For all $1 \leqslant r \leqslant q - 1$, $f_n \otimes_r f_n$ converges to zero, where $\otimes_r$ denotes the contraction of order r (see Eq. (2.5)).
(iii) $\|DF_n\|_{\mathfrak{H}}^2$ (see definition in Section 2) converges to $q\sigma^2$ in $L^2(\Omega)$ as n tends to infinity.

A new methodology to study normal approximations and to derive quantitative results combining Stein's method with Malliavin calculus was introduced by Nourdin and Peccati [15] (see also Nourdin and Peccati [16]). As an illustration of the power of this method, let us mention the following estimate for the total variation distance between the law $\mathcal{L}(F)$ of $F = I_q(f)$ and distribution $\gamma = N(0, \sigma^2)$, where $\sigma^2 = E[F^2]$:

$$d_{TV}\big(\mathcal{L}(F), \gamma\big) \leqslant \frac{2}{q\sigma^2}\sqrt{\operatorname{Var}\big(\|DF\|_{\mathfrak{H}}^2\big)} \leqslant \frac{2\sqrt{q-1}}{\sigma^2\sqrt{3q}}\sqrt{E\big[F^4\big] - 3\sigma^4}. \tag{1.1}$$