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A local inequality for Hankel operators on the sphere and its application

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Abstract

Let $H^2(S)$ be the Hardy space on the unit sphere S in $\mathbf{C}^n$. We establish a local inequality for Hankel operators $H_f = (1-P)M_f|H^2(S)$. As an application of this local inequality, we characterize the membership of H_f in the Lorentz-like ideal $\mathcal{C}_p^+$, $2n < p < \infty$.

Keywords: Hankel operator; Local inequality; Norm ideal

1. Introduction

Throughout the paper, let S denote the unit sphere $\{z \in \mathbf{C}^n: |z| = 1\}$ in $\mathbf{C}^n$. We assume that the complex dimension n is greater than or equal to 2. Let σ be the positive, regular Borel measure on S that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. As usual, the measure σ is normalized in such a way that $\sigma(S) = 1$.

Recall that the Hardy space $H^2(S)$ is the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $L^2(S, d\sigma)$. Let P be the orthogonal projection from $L^2(S, d\sigma)$ onto $H^2(S)$. Then the Hankel operator $H_f : H^2(S) \to L^2(S, d\sigma) \ominus H^2(S)$ is defined by the formula

$$H_f = (1-P)M_f|H^2(S).$$

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There is a rich literature on various kinds of Hankel operators. See, e.g., [1,2,4–6,9–13,19]. This paper falls within the so-called "one-sided" theory of Hankel operators. We remind the reader that the term "one-sided" theory refers to the study of the Hankel operator H_f alone, whereas the simultaneous study of the pair H_f and $H_{\bar f}$ is called "two-sided" theory. By virtue of the identity

$$[M_f, P] = H_f - H^*_{\bar f},$$

"two-sided" theory is equivalent to the study of the commutator $[M_f, P]$. By contrast, "one-sided" theory must deal with Hankel operators H_f that cannot be expressed in the form of $[M_g, P]$, $g \in L^2(S, d\sigma)$, which poses a much greater challenge.

Let us write $\mathbf{B}$ for the open unit ball $\{z \in \mathbf{C}^n\colon |z| < 1\}$ in $\mathbf{C}^n$. Furthermore, write

$$k_z(w) = \frac{(1-|z|^2)^{n/2}}{(1-\langle w, z\rangle)^n}, \quad |z| < 1,\ |w| \leqslant 1.$$

Then k_z is the normalized reproducing kernel for the Hardy space $H^2(S)$. In the study of Hankel operators, an extremely important role is played by the scalar quantity

$$\operatorname{Var}(f; z) = \big\| \big(f - \langle f k_z, k_z\rangle\big) k_z \big\|^2,$$

$f \in L^2(S, d\sigma)$, $z \in \mathbf{B}$. One can think of $\operatorname{Var}(f; z)$ as the "variance" of f with respect to the probability measure $|k_z|^2\, d\sigma$ on S, hence the notation.

It was shown in [16] that for $f \in L^2(S, d\sigma)$, the Hankel operator H_f is bounded if and only if $f - Pf \in \mathrm{BMO}$, which is equivalent to the boundedness of the function $z \mapsto \operatorname{Var}(f - Pf; z)$ on $\mathbf{B}$. Also, H_f is compact if and only if $f - Pf \in \mathrm{VMO}$ [16], which is equivalent to

$$\lim_{|z|\uparrow 1} \operatorname{Var}(f - Pf; z) = 0.$$

In [5] we proved that H_f belongs to the Schatten class $\mathcal{C}_p$, $2n < p < \infty$, if and only if

$$\int \operatorname{Var}^{p/2}(f - Pf; z)\, d\lambda(z) < \infty,$$

where $d\lambda$ is the standard Möbius-invariant measure on $\mathbf{B}$.

What sets the "two-sided" theory of Hankel operators apart from the "one-sided" theory is just one thing: If one has both Hankel operators H_f and $H_{\bar f}$ available, then one has the local inequality

$$\operatorname{Var}(f; z) \leqslant \|H_f k_z\|^2 + \|H_{\bar f} k_z\|^2 \tag{1.1}$$

for every $z \in \mathbf{B}$ (see [15, (6.4)]). Most of the difficulties that are particular to the "one-sided" theory of Hankel operators can be traced to the single fact that there is nothing comparable to (1.1) in the "one-sided" theory. Nothing, that is, up to this point.

One of the motivations for this paper is to find a local inequality analogous to (1.1) in the context of the "one-sided" theory of Hankel operators. This we manage to do. As it turns out, our