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Monotonicity and nonexistence results to cooperative systems in the half space

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Abstract

In this paper, we prove the monotonicity and nonexistence of positive bounded solutions to an elliptic system in the half space $\mathbb{R}^n_+$:

$$\begin{aligned} \Delta u + f(u, v) = 0 \\ \Delta v + g(u, v) = 0 \end{aligned} \quad \text{in } \mathbb{R}^n_+,$$
$$u = v = 0 \quad \text{on } \partial\mathbb{R}^n_+,$$

where $f_v \geqslant 0$, $g_u \geqslant 0$ and both f and g are convex functions.

Keywords: Monotonicity; Nonexistence; The method of moving planes

1. Introduction

In a series of seminal papers [1–4], Berestycki, Caffarelli and Nirenberg had studied positive bounded solutions in a half space $\mathbb{R}^n_+$, satisfying

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$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^n_+ = \{(x', x_n) \mid x' \in \mathbb{R}^{n-1},\ x_n > 0\}, \\ u = 0 & \text{on } x_n = 0. \end{cases} \tag{1.1}$$

In [1], they considered the problems of monotonicity and symmetry for positive solutions to Eq. (1.1). Among other things, they showed the following result.

Theorem A. *Assume $f(u)$ is Lipschitz and $f(M) \leqslant 0$, where $M = \sup u$. Then u is a function of x_n alone, and $u_{x_n} > 0$ if $x_n > 0$. Furthermore $f(M) = 0$.*

See [1] for the proof. In the same paper, the authors also proposed the following conjecture (see also [4]).

BCN conjecture. *If there is a positive bounded solution u of (1.1), then necessarily $f(M) = 0$.*

The BCN conjecture was proved by the authors themselves for $n = 2, 3$ (see [4]), and by Farina and Valdinoci [15] for $n \leqslant 5$ under some conditions on f. As far as we know, the conjecture is completely open when the dimension is bigger than 5.

Obviously, if the conjecture holds, then Eq. (1.1) should have no non-trivial positive solutions when $f(u) > 0$ for $u > 0$. In this paper, we want to consider the nonexistence of positive bounded solutions for a class of f which satisfies the following condition:

(f_1) $f \in C^1(\overline{\mathbb{R}_+}) \cap C^2(\mathbb{R}_+)$ and $f''(u) \geqslant 0$ for $u > 0$.

Theorem 1.1. *Assume that $f(0) = 0$ and (f_1) hold. If u is a nonnegative bounded solution to Eq. (1.1), then $u(x) \equiv 0$ in $\mathbb{R}^n_+$.*

Remark that the condition that u is bounded is crucial. For example, $u(x_1, \dots, x_n) = x_n(\sum_{i=1}^{n-1} e^{x_i})$ satisfies $\Delta u - u = 0$ in $\mathbb{R}^n_+$. We should mention that, if f is strictly convex with simple zeros, saying $f(u) = \lambda u + u^p$ with $\lambda \neq 0$ and $p > 1$ for instance, Theorem 1.1 follows from the nonexistence of bounded stable positive solutions of the corresponding full space problem in one lower dimension (see [10,13]) just as (1.2) below. Obviously, we can find various nonlinearities f which satisfy condition (f_1) but not strictly convex. To the best of our knowledge, Theorem 1.1 is new.

Our proof of Theorem 1.1 is based on the following monotonicity property of u, which was proved by Dancer [7,9], and by Berestycki, Caffarelli and Nirenberg [2,4].

Theorem B. *Suppose f is Lipschitz and $f(0) \geqslant 0$. If u is a positive solution of (1.1), then $u_{x_n}(x) > 0$, $\forall x \in \mathbb{R}^n_+$.*

Under the assumptions that $f \in C^1$, $f(0) \geqslant 0$ and that u is bounded, Theorem B was proved in [7,9] by the method of moving planes. Assume u is a positive bounded solution of (1.1) with $f(0) \geqslant 0$. By Theorem B, $u(x', x_n)$ is increasing in x_n. Set $u_\infty(x') = \lim_{x_n \to \infty} u(x', x_n)$, $x' \in \mathbb{R}^{n-1}$. Then u_∞ is a stable solution of

$$\Delta u_\infty + f(u_\infty) = 0 \quad \text{in } \mathbb{R}^{n-1}, \tag{1.2}$$