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A group-invariant version of Lehmer's conjecture on heights

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ABSTRACT

We state and prove a group-invariant version of Lehmer's conjecture on heights, generalizing papers by Zagier (1993) [5] and Dresden (1998) [1] which are special cases of this theorem. We also extend their three cases to a full classification of all finite cyclic groups satisfying the condition that the set of all orbits for which every non-zero element lies on the unit circle is finite and non-empty.

1. A Lehmer-type problem for the Weil height

The *Mahler measure* of a non-zero polynomial $f \in \mathbb{Z}[x]$ given by

$$f(x) = a_n \prod_{i=1}^{n} (x - \alpha_i) \tag{1}$$

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is defined as

$$M(f) = |a_n| \prod_{i=1}^{n} \max(|\alpha_i|, 1).$$

In 1933, Lehmer asked whether there exists a lower bound $D > 1$ such that for all $f \in \mathbb{Z}[x]$ it holds that

$$M(f) = 1 \quad \text{or} \quad M(f) \geq D.$$

He showed that if such a D exists, then $D \leq 1.1762808\ldots$, the largest real root of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ [4]. Nowadays, this is still the smallest known value of $M(f) > 1$ for $f \in \mathbb{Z}[x]$.

Mahler's measure is related to the Weil height of an algebraic number. Let K be an algebraic number field and v a place of K. We assume that this v-adic valuation is normalized in such a way that for all non-zero $\alpha \in K$ the product of $|\alpha|_v$ over all places v is equal to 1 and the product of $|\alpha|_v$ over all Archimedean v is equal to the absolute value of $N_{K/\mathbb{Q}}(\alpha)$. Then, for $\alpha \in K^*$ the (logarithmic) Weil height h is defined by

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v} \log^+ |\alpha|_v,$$

where the sum is over all places v of K. We used the notation $\log^+(z)$ to refer to $\log\max(z, 1)$ for $z \in \mathbb{R}$. The Weil height is independent of K and if the polynomial (1) is the minimal polynomial of α over $\mathbb{Q}$, then $h(\alpha) = \frac{1}{n} \log M(f)$. This Weil height can be extended to $\mathbb{P}^1(\overline{\mathbb{Q}})$. Namely, for $x = [x_1 : x_2] \in \mathbb{P}^1(\overline{\mathbb{Q}})$, we define

$$h(x) = \frac{1}{[K:\mathbb{Q}]} \sum_{v} \log\max(|x_1|_v, |x_2|_v),$$

where K is chosen such that $x_1, x_2 \in K$. Note that $h(x) \geq 0$.

Definition 1. Let G be a finite subgroup of $\mathrm{PGL}_2(\mathbb{Q})$. The *$G$-orbit height of* $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is defined as

$$h_G(x) = \sum_{\sigma \in G} h(\sigma x).$$

Note that $h_G(x) \geq 0$ and $h_G(\sigma\alpha) = h_G(\alpha)$ for all $\sigma \in G$. We can now state the *G-invariant Lehmer problem*, namely: given a finite group G does there exist a positive lower bound D such that

$$h_G(x) = 0 \quad \text{or} \quad h_G(x) \geq D \tag{2}$$