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On tame kernels and second regulators of number fields and their subfields ☆

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ABSTRACT

The relations between the tame kernels and second regulators of a number field F Galois over $\mathbb{Q}$, in particular the fields of degree 8 over $\mathbb{Q}$, and of some of its subfields are investigated through the action of the Galois groups on the tame kernels. In the case of the number fields with Galois group D_{2p}, a result of Browkin and Gangl is improved.

1. Introduction

For a number field F, it follows from the well-known class number formula that the value of the Dedekind zeta function $\zeta_F(s)$ at $s = 1$ (or at $s = 0$ by the function equation) has a relation with the class number $h(F)$ and the (first) regulator $R_1(F)$, while according

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to the Lichtenbaum conjecture, the value of $\zeta_F(s)$ at $s = -1$ has a relation with the $K_2\mathcal{O}_F$, the tame kernel of F, and the second regulator $R_2(F)$ (see section 2 for details).

On the other hand, Brauer [2] and Kuroda [9] have independently given the so called Brauer–Kuroda relations, that is, the multiplicative relations between the zeta functions of a number field and of some of its subfields. The Brauer–Kuroda relations have some exceptional cases (see [3,5] for details). However, in nonexceptional cases, i.e., in usually cases, the Brauer–Kuroda relations combined with the class number formula clearly can give rise to the relations between the class number of a number field and of some of its subfields (see [7]).

In the paper [4], under the assumption of the Lichtenbaum conjecture, applying the Brauer–Kuroda relations, Browkin and Gangl get formulas connecting the tame kernels and second regulators of a number field Galois over $\mathbb{Q}$ and of some of its subfields, in particular, they get the formulas in the case when the Galois group of F equals $\mathbb{Z}/2\times\mathbb{Z}/2$ or D_{2p}, the dihedral group with p an odd prime number, or the alternating group A_4.

In the present paper, we consider the case when the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ equals $\mathbb{Z}/4\times\mathbb{Z}/2$, the dihedral group D_8, and we also consider the case when the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ equals the dihedral group D_{2p}. In particular, we improve the result of [4] about the case of D_{2p}, that is, we prove the equality of orders of tame kernels holds for the part not divided by 2 and p without the assumption of the Lichtenbaum conjecture.

2. The value of $\zeta_F^*(-1)$ and the second regulator

Let F be a number field and $\zeta_F(s)$ the Dedekind zeta function of F. It is well-known that $\zeta_F(s)$ is a meromorphic function on $\mathbb{C}$ with a unique single pole at $s = 1$. It has zeros in the strip $\{s \in \mathbb{C} : 0 < \mathrm{Re}s < 1\}$ and at most at nonpositive integers $-m, m \geq 0$. The multiplicity of zero at $s = -m$ is

$$d_m = \begin{cases} r_1 + r_2 - 1 & \text{if } m = 0, \\ r_1 + r_2 & \text{if } m \text{ is even}, m > 0, \\ r_2 & \text{if } m \text{ is odd}, \end{cases}$$

where r_1 is the number of real places of F, and r_2 is the number of complex ones. We have $[F : \mathbb{Q}] = r_1 + 2r_2$.

We are interested in the case $r_1 = 0$, so $\zeta_F(-1) = 0$. Let $\zeta_F^*(-1)$ be the first nonzero coefficient in the Taylor expansion of $\zeta_F(s)$ in a neighborhood of $s = -1$.

The second regulator is related with the Bloch group of F, which is defined by the dilogarithm of Bloch and Wigner.

Now, following Browkin and Gangl [4], we define the Bloch–Wigner function by

$$\widetilde{D}(z) := -\mathrm{Im}\Big(\frac{1}{\pi}\int_0^z \frac{\log(1-t)}{t}dt\Big) + \frac{\arg(1-z)}{\pi}\cdot \log|z|,\ \ z \in \mathbb{C}.$$