Journal of Number Theory 171 (2017) 301–340

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Series expansion of the period function and representations of Hecke operators

Dohoon Choi [a], Subong Lim [b,*], Tobias Mühlenbruch [c], Wissam Raji [d]

[a] *School of liberal arts and sciences, Korea Aerospace University, 200-1, Hwajeon-dong, Goyang, Gyeonggi 412-791, Republic of Korea*
[b] *Department of Mathematics Education, Sungkyunkwan University, 25-2, Sungkyunkwan-Ro, Jongno-gu, Seoul 03063, Republic of Korea*
[c] *Department of Mathematics and Computer Science, FernUniversität in Hagen, 58084 Hagen, Germany*
[d] *Department of Mathematics, American University of Beirut, Beirut, Lebanon*

ARTICLE INFO

Article history:
Received 25 February 2016
Received in revised form 2 July 2016
Accepted 3 July 2016
Available online 7 September 2016
Communicated by A. El-Guindy

MSC:
primary 11F37
secondary 20C08, 32N10, 33B20

Keywords:
Period functions
Eichler–Shimura cohomology
Hecke operators

ABSTRACT

The period polynomial of a cusp form of an integral weight plays an important role in the number theory. In this paper, we study the period function of a cusp form of real weight. We obtain a series expansion of the period function of a cusp form of real weight for $\mathrm{SL}(2,\mathbb{Z})$ by using the binomial expansion. Furthermore, we study two kinds of Hecke operators acting on cusp forms and period functions, respectively. With these Hecke operators we show that there is a Hecke-equivariant isomorphism between the space of cusp forms and the space of period functions. As an application, we obtain a formula for a certain L-value of a Hecke eigenform by using the series expansion of its period function.

* Corresponding author.
E-mail addresses: choija@kau.ac.kr (D. Choi), subong@skku.edu (S. Lim), tobias.muehlenbruch@fernuni-hagen.de (T. Mühlenbruch), wr07@aub.edu.lb (W. Raji).

http://dx.doi.org/10.1016/j.jnt.2016.07.020
0022-314X/

1. Introduction

For an even integer $k \geq 2$, denote by S_k the space of cusp forms of weight k and the trivial multiplier system on $\mathrm{SL}(2,\mathbb{Z})$. For each cusp form $f \in S_k$, the period polynomial of f is defined by

$$\int_0^{i\infty} f(\tau)(\tau - z)^{k-2}\,\mathrm{d}\tau.$$

Then the Eichler–Shimura cohomology theory tells us that there is an isomorphism between the direct sum $S_k \oplus S_k$ and a subspace of the vector space

$$W_k = \left\{P \in P_{k-2};\ P + P\big|_{2-k}S = P + P\big|_{2-k}U + P\big|_{2-k}U^2 = 0\right\}$$

with codimension 1. Here, P_{k-2} is the space of polynomials of degree at most $k-2$, S, T and U denote the matrices

$$S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{and} \quad U := \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

in $\mathrm{SL}(2,\mathbb{Z})$ and $\big|_{2-k}$ is the slash operator as in (2.2). We refer to [Lan] for more details on this isomorphism. The coefficients of the period polynomial are also called periods. These periods give two additional rational structures on S_k besides the usual rational structure given by the rationality of Fourier coefficients. Periods are also related with the critical values of L-functions and have been much studied by many researchers (for example, see [CZ,Kno2,KZ,Man,Za]). Another related objects are the Hecke operators acting on cusp forms. On S_k, the mth Hecke operator is represented with triangle matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ of non-negative integer entries satisfying $ad = m$ and $0 \leq b < d$ which act on f by extending the slash operator accordingly. Their representation on period polynomials were also studied (for example, see [CZ,Man,PP]).

In this paper, we consider cusp forms of real weight. In general the associated period function is not a polynomial any more even when the weight is half-integral. This is a big difference between the cases of integral and real weight. We discuss some aspects of such period functions in this paper. More precisely, we obtain a series expansion of the period function of a cusp form of real weight for $\mathrm{SL}(2,\mathbb{Z})$ by using the binomial expansion. The coefficients of this expansion can be expressed in terms of the incomplete gamma function. Furthermore, we study two kinds of Hecke operators acting on cusp forms and period functions, respectively. With these Hecke operators we show that there is a Hecke-equivariant isomorphism between the space of cusp forms and the space of period functions. As an application, we obtain an exact formula for a certain L-value of a Hecke eigenform by using the series expansion of its period function.

Throughout this paper let $k \in \mathbb{R}$ be a real weight with compatible multiplier system χ. We denote by $S_{k,\chi}(N)$ the space of cusp forms of weight k and multiplier system χ on $\Gamma_0(N)$. If $f \in S_{k,\chi}(1)$, then f has a Fourier expansion of the form