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A modular interpretation of various cubic towers

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ABSTRACT

In this article we give a Drinfeld modular interpretation for various towers of function fields meeting Zink's bound.

1. Introduction

Let p be a prime number and $q = p^n$ for a positive integer n. It is a central question in algebraic geometry how many $\mathbb{F}_q$-rational places $N(F)$ a function field F with full constant field $\mathbb{F}_q$ and genus $g(F)$ can have. For small genus the Hasse–Weil estimate $N(F) \leq q + 1 + 2g(F)q^{1/2}$ is good, but the larger the genus compared to the size of the finite field $\mathbb{F}_q$ it gets worse. For this reason, Ihara introduced the constant

$$A(q) := \limsup_{g(F)\to\infty} \frac{N(F)}{g(F)} ,$$

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where the limit is taken over all function fields F with full constant field $\mathbb{F}_q$ and genus tending to infinity. It is known that $0 < A(q) \leq \sqrt{q} - 1$, the first inequality being due to Serre [15], while the second inequality is known as the Drinfeld–Vladut bound [16]. Combining the work of Ihara [13] and the Drinfeld–Vladut bound, one sees that $A(q) = \sqrt{q} - 1$ if q is square; i.e. n is even.

Ihara used reductions of modular curves to obtain his result. It was a surprise when in [8] completely different methods were used to prove the same result. In [8] a lower bound for $A(q)$ is obtained using a *tower (of function fields) over* $\mathbb{F}_q$,

$$\mathcal{F} = (F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_i \subseteq \ldots) .$$

It is required that all function fields F_i have full constant field $\mathbb{F}_q$, and $g(F_i) \to \infty$ as $i \to \infty$. Also all extensions F_{i+1}/F_i are assumed to be separable. These assumptions imply that the following limit exists:

$$\lambda(\mathcal{F}) := \lim_{i\to\infty} \frac{N(F_i)}{g(F_i)} ,$$

which is called the limit of the tower $\mathcal{F}$. One then obtains the lower bound for Ihara's constant: $A(q) \geq \lambda(\mathcal{F})$. The main ingredient in [8] was to explicitly produce a tower of function fields over $\mathbb{F}_q$ with limit $q^{1/2} - 1$. This method also turned out to be fruitful for nonsquare q. For $p = 2$, using explicit towers of function fields, it was shown in [9] that $A(8) \geq 3/2$, while this result was generalized in [5] to

$$A(q^3) \geq \frac{2(q^2 - 1)}{q + 2} . \tag{1}$$

The generalization was achieved by explicitly constructing a tower of function fields over $\mathbb{F}_{q^3}$ (which we will call a *cubic* tower) with limit $\lambda(\mathcal{F}) \geq 2(q^2 - 1)/(q + 2)$. Since then several other papers have appeared in which other towers or alternative descriptions of previously known towers were formulated, giving rise to various cubic towers with the same limit [14,6,2]. Note that these results generalize the statement by Zink in [17] that $A(p^3) \geq 2(p^2 - 1)/(p + 2)$ for a prime p. For this reason the bound in Equation (1) is called Zink's bound.

The different methods of using explicit towers on the one hand and reductions of modular curves on the other hand became interlinked when Elkies showed that the towers in [8] can also be obtained using the theory of Drinfeld modules in [7]. Despite the recent developments in [2] and [11], where the theory of Drinfeld modules and their moduli spaces was used to construct sequences of curves with many rational points over any non-prime field, it remained a mystery to what extent the original cubic towers meeting Zink's bound, can be explained from the modular theory. In this article we solve this problem.