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Rational digit systems over finite fields and Christol's Theorem ☆

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A B S T R A C T

Let $P, Q \in \mathbb{F}_q[X] \setminus \{0\}$ be two coprime polynomials over the finite field $\mathbb{F}_q$ with $\deg P > \deg Q$. We represent each polynomial w over $\mathbb{F}_q$ by

$$w = \sum_{i=0}^{k} \frac{s_i}{Q} \left(\frac{P}{Q}\right)^i$$

using a rational *base* P/Q and *digits* $s_i \in \mathbb{F}_q[X]$ satisfying $\deg s_i < \deg P$. *Digit expansions* of this type are also defined for formal Laurent series over $\mathbb{F}_q$. We prove uniqueness and automatic properties of these expansions. Although the ω-language of the possible digit strings is not regular, we are able to characterize the digit expansions of algebraic elements.

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In particular, we give a version of Christol's Theorem by showing that the digit string of the digit expansion of a formal Laurent series is automatic if and only if the series is algebraic over $\mathbb{F}_q[X]$. Finally, we study relations between digit expansions of formal Laurent series and a finite fields version of Mahler's 3/2-problem.

1. Introduction

We study digit systems with "rational bases" defined in rings of polynomials and fields of formal Laurent series over finite fields. Although the ω-language of the digit strings of expansions with respect to such a digit system is not regular, expansions of algebraic elements are well-behaved. In particular, we are able to establish a version of Christol's Theorem for expansions of formal Laurent series.

Digit systems over finite fields have been first studied in 1991 by Kovács and Pethő [14]. Since then they have been generalized in various ways (see [7,19,22,25]) and many of their properties have been investigated (*cf.* [1,6,8,15–17,24]). As indicated in Effinger et al. [12] theories sometimes reveal new features when transferred from the integer to the finite fields setting. This is true also for digit systems over finite fields: although they often behave similar to their "cousins" defined over $\mathbb{Z}$ and $\mathbb{R}$, they also show completely different properties that have no analogue in the integer case. This is highlighted for instance by the p-automaticity results by Rigo [19], Rigo and Waxweiler [20] and, more recently, by Scheicher and Sirvent [23]. The specialty of the digit systems studied in the present paper consists in the fact that they have rational functions as bases and, hence, form the analogues of the rational based digit systems studied by Akiyama et al. [3].

The rational based digit systems studied by Akiyama et al. [3] have a very difficult structure and are related to the notorious 3/2-problem of Mahler [18]. The present contribution aims at getting more information on the structure of their finite fields analogues. Although some properties that we face are similar to the ones in the integer setting, we put emphasis in proving results that are special to the finite fields setting. These new results are often related to automaticity properties of the digit systems over finite fields. As mentioned above, one of our main results is a version of Christol's Theorem (*cf.* Theorem 7.2). To prove this we derive explicit formulas for the digit expansion of a given formal Laurent series (see Theorem 6.7). A motivation of these results comes from the fact that they contrast the results by Adamczewski and Bugeaud [2] stating that in the case of q-ary number systems automaticity of expansions gives rise to transcendental numbers. Indeed, in the case of our Laurent series expansions automaticity of expansions yields algebraicity of the expanded element.

To be more precise, let P and Q be nonzero and coprime polynomials over a finite field $\mathbb{F}_q$ whose degrees satisfy $\deg P > \deg Q$. Then by a simple algorithm we can expand each polynomial w over $\mathbb{F}_q$ by its *P/Q-polynomial digit expansion*